Phil 5312
Fall 2024

Assignment 7:
You may do this homework assignment in lieu of a final paper if you wish. It will be due Friday morning, December 13th.

Choose any two parts of this homework and do every problem in each of those two parts.

Part 1: Probability problems
Do the following problems in Titelbaum, *Foundations of Bayesian Epistemology*. It is likely you will need to read at least portions of the chapters.

2.5, 2.7, 2.9, 3.1, 3.3, 3.5, 3.7

In addition, do the following two problems:

1) Assume that Pr is a probability function that satisfies the following:
P[E] = 0.55
P[F] = 0.5
P[¬G] = 0.45
P[E & F]=0.3
P[E & ¬G]=0.25 P[F & ¬G]=0.3
P[E & F & ¬G]=0.2
Find P[¬E & ¬F & G]. Show your work in a way that makes it clear how you could have done a slightly different problem. For example, if you use a Venn diagram, tell me the order you filled in the regions and which regions correspond to the answer. If you use the algebraic method, just write the relevant equations, etc.).

2) Define the false positive rate of a test to be the probability of getting a positive result given that the patient does not have the disease. The false negative rate is the probability of getting a negative result given that they do have it.

You overhear a doctor tell her patient: "Now after the last set of tests, I told you I was 75% sure that you had the antibody in your blood so we decided to do another test. Well, now I can say that I am 95% sure. After all, the test came out positive and the false positive rate on this test is only 10%." Now you know that this doctor is a competent statistician. What can you infer about the false negative rate of the test? If the test had instead come out negative, how should the doctor have revised her beliefs?

Part 2: Probabilities and Conditionals

Reading Bennett chapter 9 may help you here.

Prove that each of the following holds for any probability function P and any propositions A, C where P(A) > 0 (so that P(C|A) is defined)

1) $P(C|A) \leq P(A \supset C)$.
2a) if $P(A) = 1$, then $P(C|A) = P(A \supset C) = P(C)$
2b) $P(C|A) = 1$ if and only if $P(A \supset C) = 1$
2c) $P(C|A) = P(A \supset C)$ entails that one of these two cases above obtains (that is, entails that either $P(A) = 1$ or $P(A \supset C) = 1$)

In problems 3-7 below, for a given material conditional $A \supset C$, call $P(C|A)$ 'the corresponding conditional probability'

For each of arguments 3-7, say whether they are deductively valid. Now replace any material conditionals with the corresponding conditional probability. Now assume that the probability of each of the premises is 1. What is the possible range of the probability of the conclusion? Next, make the probability of the premises each .9. Now what is the possible range of the probability of the conclusion? Is the argument probabilistically valid?

3) $A \supset C, A \vdash C$
4) $A \supset C, C \vdash A$
5) $C \vdash A \supset C$
6) $A \supset C \vdash (A\&B) \supset C$
7) $A \supset (C\&B) \vdash A \supset C$

Part 3: Counterfactuals

1) In "Most Counterfactuals are False", Alan Hájek describes a principle that he calls "The Poisoning Principle":

PP: The disjunctive-antecedent counterfactual
$(D_1 \text{ or } D_2 \text{ or } \ldots \text{ or } D_n) > C$ is false
if any of the individual-disjunct counterfactuals
$D_i > C$ is false.

Bennett talks about the principle SDA (Simplification of disjunctive antecedents):

SDA: $(A \vee B) > C \vdash (A > C) \wedge (B > C)$

Another important principle is antecedent strengthening:

AS: $A > C \vdash (A \wedge B) > C$

Show that these three principles are all equivalent to each other (by a chain of equivalences).

You may assume that if one proposition is logically equivalent to another, you can freely substitute it in a formula and the truth-value will be the same. For example, $A \vee B$ is equivalent to $B \vee A$ and so anything that is true with '$A \vee B$' in it will also be true if we replace '$A \vee B$' with '$B \vee A$'.

Hint: A is logically equivalent to $(A \wedge B) \vee (A \wedge \neg B)$

2) Show that all three principles are invalid in the Lewis/Stalnaker possible worlds semantics (since they are equivalent, show any one is invalid).

3) Show that all three principles are valid if the counterfactual is really a strict conditional - that is, if $A > C$ means $\Box(A \supset C)$. (Showing one is valid in modal logic K would suffice).

4) This might seem like a reasonable view of the counterfactual: $A > C$ is true when at the nearest A world(s) the probability of C is very high. Take 'very high' to mean $\geq 2/3$ here. One argument against this kind of analysis is that it would violate the principle that $A > B$ together with $A > C$ entails $A > (B \wedge C)$. Explain why this analysis would violate this principle. Also explain why the number 2/3 here is arbitrary in the sense that any probability < 1 would have exactly the same problem.

5) One key difference between Lewis's and Stalnaker's semantics is that Stalnaker assumes that there is a unique 'closest' A-world (if there is one at all). For many inferences, this does not make a difference. But it does for these two cases below. For each inference, explain whether they are valid or not on the Lewis/Stalnaker semantics and then give an argument that the inference is either *really* valid or *really* invalid (an example would probably be a good way to do this).

5a. $A > (B \vee C), \neg (A > B)$ therefore $A > C$
5b. $\forall x \neg (A > Fx)$ therefore $A > \forall x \neg Fx$

6) Assume the basic Lewisian semantics for counterfactuals: $A>C$ is true iff there is no possible world where A is true or there is a possible world where $A \wedge C$ is true which is closer to the actual world than any $A \wedge \neg C$ world. This leaves 'closer' undefined. Each of 1-5 is a proposed analysis of 'closer' which Lewis would reject. Give an example counterfactual which you think is clearly true (or clearly false), but which the proposed analysis would count as false (or true) and explain why the analysis does so. Or alternatively, give some other argument for why this is an unacceptable analysis.

6.1) If two worlds both have human beings in them then they are automatically closer to each other than a world that does and a world that doesn't.

6.2) Assuming that the relevant worlds all have exactly the same people in them, any two worlds where everybody lives to be exactly the same age in the two worlds (so Joel dies at the same age in both, Bob dies at the same age in both, etc.) are automatically closer to each other than worlds that aren't like this.

6.3) A world w is closer to y than to z iff they share more facts in common

6.4) Any two worlds which share the exact same past (before the time when A occurred) are closer to each other than either is to a world which doesn't share the exact same past.

6.5) Any worlds which share exactly the same causal laws (laws of nature) are closer to each other than any two that don't.

Part 4: Modal logic
Using MacFarlane's setup, here is a complete set of rules for Modal Logic K: propositional logic rules, MNE, □Intro, Modal Reit - T (yes, it is poorly named).

Give a natural deduction showing that the following are valid in K:

1) $\Box(P \supset Q) \vdash \Diamond(P \wedge R) \supset \Diamond(Q \wedge R)$
2) $\Diamond P \vdash \Diamond(Q \vee \neg Q)$
3) $\vdash \Diamond(P \supset Q) \equiv (\Box P \supset \Diamond Q)$

For each of these sentences, determine whether or not they are logical truths in T, S4, and S5. If logical truths, give a deduction. If not, give a countermodel.

4) $P \supset (\Diamond\Box\Box P \supset \Box P)$
5) $\Box(\Box P \vee \Box Q) \equiv (\Box P \vee \Box Q)$
6) $\Diamond\neg P \vee \Diamond\neg Q \vee \Diamond(P \wedge Q)$

7) Give an argument that there are no logical truths of K of the form $\Box\Diamond\phi$.

8) Give an argument that if $\phi \supset \psi$ is a logical truth of K, then $\Box\phi \supset \Box\psi$ will also be a logical truth in K.

Correspondence of axioms and frames:
Recall that a frame is a non-empty set of worlds with an accessibility relation between worlds. So it *part* of a model that doesn't have any truth value assignments to variables. A formula is said to be *valid* on a frame if will turn out true at every world in that frame regardless of the truth-value assignment to the atomic sentences.

A formula is said to correspond to a condition on frames if it is valid on all and only frames that satisfy that condition. For example, $\Box P \supset P$ corresponds to reflexivity ($\forall$wRww) because if every world can see itself in a given frame then $\Box P \supset P$ will be true at every world and if any world cannot see itself, then it is possible to construct a model where $\Box P \supset P$ turns out false at a world (pick a world that doesn't see itself, make P false there and make P true everywhere else).

Some of the more commonly studied frame conditions are listed in section 8 here: https://plato.stanford.edu/entries/logic-modal/

One condition that Garson calls "functional" is the frame condition $\forall w \forall v \forall u$ ($Rwv \wedge Rwu) \supset v = u$). In other words, each world can see at most one world.

9) Prove that $\Diamond P \supset \Box P$ is valid on a frame iff that frame satisfies this "functional" condition. (So first assume every world can see at most one world and prove the formula will be true everywhere and next assume the frame doesn't satisfy the condition and prove there is a countermodel).

10) Another important condition is "convergence":
$\forall w \forall v \forall u ((Rwv \wedge Rwu) \supset \exists x(Rvx \wedge Rux))$

This is called "convergence" because if you have a world that can see two possibilities, they will "converge" back together and not stay separate. (Think of time branching and then converging back to a single future).

Prove that $\Diamond\Box P \supset \Box\Diamond P$ corresponds to the convergence frame condition.

HINT:
Here is a sample, very complete answer. $\Box P \supset \Box\Box P$ corresponds to the frame condition $\forall w \forall v \forall u((Rwv \wedge Rvu) \supset Rwu)$. Proof:

Assume that the frame is transitive. Now pick an arbitrary world w in that frame. Assume that $\Box P$ is true at w. That means that P is true at every world that w can see. Now for reductio, assume that $\Box\Box P$ is false at w. That means that there is a world v where Rwv and where $\Box P$ is false at v. That means that there is a world u where Rvu and where P is false at u. So now we have Rwv and Rvu and so by the frame condition (transitivity) we have Rwu. Since Rwu and $\Box P$ is true at w, we now have P is true at u. This is a contradiction. So $\Box\Box P$ must be true at w. So therefore $\Box P \supset \Box\Box P$ is true at w. w was arbitrary so $\Box P \supset \Box\Box P$ must be true at every world in the frame. This was an arbitrary transitive frame, so $\Box P \supset \Box\Box P$ must be valid in every transitive frame.

Now assume that the frame is NOT transitive. That means $\forall w \forall v \forall u((Rwv \wedge Rvu) \supset Rwu)$ is false which means $\exists w \exists v \exists u(Rwv \wedge Rvu \wedge \neg Rwu)$. We will now construct a countermodel:

Make P true at every world in the frame except for u. Now since Rwu is false and P is true everywhere except u, $\Box P$ will be true at w. However, we have Rvu and P is false at u, therefore $\Box P$ is false at v. Now since Rwv and $\Box P$ is false at v, we have $\Box\Box P$ is false at w. Therefore $\Box P$ is true at w and $\Box\Box P$ is false at w so $\Box P \supset \Box\Box P$ is false at w. So $\Box P \supset \Box\Box P$ is not valid in this particular frame. This frame was totally arbitrary (except that it didn't satisfy transitivity) so therefore $\Box P \supset \Box\Box P$ must not be valid in any frame that is not transitive.