Philosophy 5311: Bayesian Epistemology
Handout for Sept 3, 2014

Beginning on page 23 of *Foundations of Bayesian Epistemology*, Titelbaum lists several fairly quick and very useful consequences of the probability axioms. With notational variants, two of them are:

Entailment: For any propositions P and Q in L, if $P \vDash Q$ then $Pr(P) \leq Pr(Q)$

Equivalence: For any propositions P and Q in L, if $P \Dashv\vDash Q$ then $Pr(P) = Pr(Q)$.

In class I attempted to prove these from the axioms by first proving entailment and then using that to prove equivalence. But the proof I tried to use for entailment just presupposed equivalence. So that is no good. What we need is to prove one of these without presupposing the other.

Here is a proof of equivalence I eventually came up with (with help from the great internet…)

1) Assume that $P \Dashv\vDash Q$
2) $P \vee \neg Q$ is a tautology (this follows from 1)
3) $Pr(P \vee \neg Q) = 1$ (follows from axiom 2)
4) $P \vDash \neg\neg Q$ (from 1)
5) $Pr(P \vee \neg Q) = Pr(P) + Pr(\neg Q)$ (from axiom 3 – given line 4)
6) $1 = Pr(P) + Pr(\neg Q)$ (from 3 and 5)
7) $1 - Pr(\neg Q) = Pr(Q)$ (negation – proved earlier in class)
8) $Pr(P) = Pr(Q)$ (from 6 and 7)

Now that we have proved equivalence, we can go ahead and prove entailment the way that I originally did:

1) Assume that $P \vDash Q$
2) $P \Dashv\vDash (P\&Q) \vee (P\&\neg Q)$ (fact of logic)
3) $Pr(Q) = Pr((P\&Q) \vee (\neg P\&Q))$ (equivalence with line 2)
4) $Pr(Q) = Pr(P\&Q) + Pr(\neg P\&Q)$ (axiom 3 – since P&Q and ¬P&Q are exclusive)
5) $P \Dashv\vDash P\&Q$ (from 1)
6) $Pr(P) = Pr(P\&Q)$ (equivalence plus line 5)

7) $Pr(Q) = Pr(P) + Pr(\neg P\&Q)$ (from lines 4 and 6)
8) $Pr(\neg P\&Q) \geq 0$ (axiom 1)
9) $Pr(Q) \geq Pr(P)$ (from 7 and 8)