Monty Hall Problem
Handout for Phil 5311

Wikipedia has a very lengthy article with some interesting history and lots of details about the Monty Hall Problem. It is well worth reading.

http://en.wikipedia.org/wiki/Monty_Hall_problem

I said in class that under certain assumptions (Monty always opens an empty door and chooses randomly if he has a choice) then you ought to switch. I also said that if all you learned was that it was not behind a certain door, then it makes no difference if you switched. Let's go through these two cases carefully.

First, lets call the hypothesis that the prize is behind door 1, D1. So that fact that it is not behind door 2 would be ~D2 for example. Call the fact that Monty revealed door 2 to be empty after you chose door 1 OD2.

Now lets imagine you initially choose D1 (in class we said we choose D3 so make sure to notice the transition if you are comparing your notes to this handout). Now before Monty does anything, we have a probability distribution for the prizes being behind different doors P(D1), P(D2), and P(D3). Lets assume for this problem that these are each equal to 1/3.

First, let's imagine all we learned is that in fact the prize is not behind door 2. That is, we learned ~D2. What should our new credence be that the prize is behind door 1 vs. door 3? In this case, conditionalizing on ~D2 will lead us to P(D1) = 1/2. That is, P(D1|~D2) = 1/2. Here is a proof:

$P(D1|{\sim}D2) = P({\sim}D2|D1) \times P(D1)/P({\sim}D2) = (1 \times 1/3) / 2/3 = 1/2$. This is pretty straightforward.

Now imagine that after choosing door 1, Monty reveals that door 2 is empty. We showed in class that in that case, you should switch. Part of understanding the answer that you should switch in the game is therefore figuring out how you learn anything relevant beyond just that it isn't behind door 2.

Of course what you really learned is that Monty opened door 2 after you selected door 1. But why is conditioning on this different than conditioning on just ~D2? Because there are different scenarios in which he could open door 2 and the probabilities and likelihoods for these scenarios are different.

This is what you should believe:

You have selected door 1 and Monty has not yet revealed anything. We will call this t1 at which point you have some probability function P_1. Now after t1 Monty then reveals that door 2 is empty. Now you have function P_2. We assume that you should update by conditionalization so $P_2(D1) = P_1(D1|OD2)$.

By Bayes' Theorem $P_1(D1|OD2) = P_1(OD2|D1) \times P_1(D1)/P_1(OD2)$. By the law of total probability, $P_1(OD2) = P_1(OD2|D1) \times P(D1) + P_1(OD2|D2) \times P(D2) + P_1(OD2|D3) \times P(D3)$.

The assumption that Monty always opens an empty door and chooses randomly if he has a choice means that:

$P_1(OD2|D1) = 1/2$
$P_1(OD2|D2) = 0$
$P_1(OD2|D3) = 1$

Add the assumption that $P_1(D1) = P_1(D2) = P_1(D3) = 1/3$ and we get the standard answer that $P_2(D1) = 1/3$ and that $P_2(D3) = 2/3$. So you should switch.

Here is an older puzzle from Joseph Bertrand's *Calcul des probabilités* – the same book that introduces the chord problem (which will be mentioned later in class).

> There are three boxes:
>
> 1. a box containing two gold coins
> 2. a box with two silver coins
> 3. a box with one gold and one silver coin
>
> You select a box at random and pick a random coin from that box. It is gold. What is the probability that the other coin in the box is also gold?

There is an obvious argument for the answer of 1/2. There were three boxes and now you know for sure that it is not in box 2. Therefore it is either in box 1 or box 3 and so the probability is 1/2. However, this is bad reasoning. In fact, the probability is 2/3 which you can calculate by using Bayes' Theorem. The key asymmetry is that the probability that you select a gold coin after picking box 1 is higher than if you pick box 3. Notice that if, for example, the coins were labeled 'coin 1' and 'coin 2' and you picked coin 1 and it was gold, now 1/2 would be the correct answer. This trick of labeling is related to the extremely confusing 'boy or girl paradox' which you can read about on wikipedia here: http://en.wikipedia.org/wiki/Boy_or_Girl_paradox

There are obvious formal features of this problem and others (like the 'three prisoner's problem') which make some people say that 'Bertrand's box problem' is logically equivalent to the Monty Hall puzzle. I disagree. The numerical answers are the same for similar reasons, but they are not the same puzzle.