

Handout for Phil 5311

Definition: A gamble on A made at odds of a:b for stakes of \$S is such that it pays $\$bS/(a+b)$ if A is true and $-\$aS/(a+b)$ if A is false. (i.e. if you are betting with a bookie, you pay \$a to place a bet where you would win back $\$(a+b)$ and net \$b in winnings if you win the bet.) Since the stakes are arbitrary, odds of a:b are the same as odds of ax : bx for any x.

Definition: A conditional gamble on A given B at odds of a:b at stakes \$S is such that it pays $\$bS/(a+b)$ if A&B is true, pays $-\$aS/(a+b)$ if $(\sim A)\&B$, and pays 0 if $\sim B$ (i.e. the bet is called off).

Definition: An agent considers a gamble fair if the subjective expected utility of the gamble is \$0.

Definition: An agent's betting quotient (or Degree of Belief) for a proposition A [abbreviated DoB(A)] is the value q such that the agent considers a bet at price \$q for stake \$1 to be fair.

-- Assuming that the agent values money linearly, [so that $U(\$x) + U(\$y) = U(\$x+y)$], this means that if the agent considers a bet on A at odds a:b fair, then the agent's betting quotient is $a/(a+b)$.

Conditional betting:

Lets call $\text{DoB}(A\&B) = q$

Lets call $\text{DoB}(B) = r$

For simplicity and clarity, we can assume here $\text{DoB}(\sim B) = 1-r$

A conditional bet on A given B at price $\$ \text{DoB}(A\&B) / \text{DoB}(B)$ to win \$1 has the following payoff table:

	Total payoff in dollars = winnings minus amount paid for bet
A & B	$1-(q/r)$
A & $\sim B$	0
$\sim A$ & B	$-q/r$
$\sim A$ & $\sim B$	0

We can simulate a conditional bet on A given B by making two unconditional bets – one on A&B and one on $\sim B$.

In order to come up with the two bets, we aim to have the same payoff table as above. We are going to make a bet on A&B and a bet on $\sim B$. The easiest way to make sure we have the right payoff table is to aim to have 0 total payoff in the $\sim B$ cases.

Since we will lose the A&B bet, we make sure that the total net winnings on the $\sim B$ bet exactly cancel out the cost of placing the A&B bet.

The following theorem makes this easier to see how to do:

Theorem: $P(A|B) = P(A\&B) + P(\sim B) \times P(A\&B)/P(B)$

Thus if we wanted to simulate a bet on A given B with stakes of \$1, we could use a bet on A&B stakes of \$1 and a bet on $\sim B$ at stakes of $\$DoB(A\&B)/DoB(B) = \q/r

Bet 1: Bet on A&B at \$q to win \$1 – this is odds $q:1-q$

Bet 2: Bet on $\sim B$ at $\$(q/r) \times (1-r)$ to win $\$q/r$ – this is odds $(q/r) \times (1-r) : (q/r) - (q/r) \times (1-r)$ which is the same as odds $(1-r) : r$

Now the results of making these two bets together:

	Payoff bet 1	Payoff bet 2	Total Payoff
A & B	$1-q$	$-(q/r) \times (1-r)$	$1-(q/r)$
A & $\sim B$	$-q$	$(q/r) - [(q/r) \times (1-r)] = q$	0
$\sim A$ & B	$-q$	$-(q/r) \times (1-r)$	$-q/r$
$\sim A$ & $\sim B$	$-q$	$(q/r) - [(q/r) \times (1-r)] = q$	0

Notice that the payoff table for making both of these bets is exactly the same as making a single conditional bet. Therefore, if you value the single conditional bet at a different rate than you value the sum of these two bets, you can be Dutch Booked.