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ON INDUCTIVE LOGIC

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§1. INDUCTIVE LOGIC

Among the various meanings in which the word 'probability' is used in everyday language, in the discussion of scientists, and in the theories of probability, there are especially two which must be clearly distinguished. We shall use for them the terms 'probability$_1$' and 'probability$_2$'. Probability$_1$ is a logical concept, a certain logical relation between two sentences (or, alternatively, between two propositions); it is the same as the concept of degree of confirmation. I shall write briefly "$\mathfrak{c}$" for "degree of confirmation," and "$\mathfrak{c}(h, e)$" for "the degree of confirmation of the hypothesis h on the evidence e"; the evidence is usually a report on the results of our observations. On the other hand, probability$_2$ is an empirical concept; it is the relative frequency in the long run of one property with respect to another. The controversy between the so-called logical conception of probability, as represented e.g. by Keynes[1], and Jeffreys[2], and others, and the frequency conception, maintained e.g. by v. Mises[3] and Reichenbach[4], seems to me futile. These two theories deal with two different probability concepts which are both of great importance for science. Therefore, the theories are not incompatible, but rather supplement each other.[5]

In a certain sense we might regard deductive logic as the theory of L-implication (logical implication, entailment). And inductive logic may be construed as the theory of degree of confirmation, which is, so to speak, partial L-implication. "e L-implies h" says that h is implicitly given with e, in other words, that the whole logical content of h is contained in e. On the other hand, "$\mathfrak{c}(h, e) = 3/4$" says that h is not entirely given with e but that the assumption of h is supported to the degree 3/4 by the observational evidence expressed in e.

In the course of the last years, I have constructed a new system of inductive logic by laying down a definition for degree of confirmation and developing a theory based on this definition. A book containing this theory is in preparation. The purpose of the present paper is to indicate briefly and informally the definition and a few of the results found; for lack of space, the reasons for the choice of this definition and the proofs for the results cannot be given here. The book will, of course, provide a better basis than the present informal summary for a critical evaluation of the theory and of the fundamental conception on which it is based.[6]

[1] J. M. Keynes, *A Treatise on Probability*, 1921.

[2] H. Jeffreys, *Theory of Probability*, 1939.

[3] R. v. Mises, *Probability, Statistics, and Truth*, (orig. 1928) 1939.

[4] H. Reichenbach, *Wahrscheinlichkeitslehre*, 1935.

[5] The distinction briefly indicated here, is discussed more in detail in my paper "The Two Concepts of Probability," which will appear in *Philos. and Phenom. Research*, 1945.

[6] In an article by C. G. Hempel and Paul Oppenheim in the present issue of this journal, a new concept of degree of confirmation is proposed, which was developed by the two authors and Olaf Helmer in research independent of my own.

§2. SOME SEMANTICAL CONCEPTS

Inductive logic is, like deductive logic, in my conception a branch of semantics. However, I shall try to formulate the present outline in such a way that it does not presuppose knowledge of semantics.

Let us begin with explanations of some semantical concepts which are important both for deductive logic and for inductive logic.[7]

The system of inductive logic to be outlined applies to an infinite sequence of finite language systems L_N ($N = 1, 2, 3$, etc.) and an infinite language system L_∞. L_∞ refers to an infinite universe of individuals, designated by the individual constants 'a_1', 'a_2', etc. (or 'a', 'b', etc.), while L_N refers to a finite universe containing only N individuals designated by 'a_1', 'a_2', $\cdots$ 'a_N'. Individual variables 'x_1', 'x_2', etc. (or 'x', 'y', etc.) are the only variables occurring in these languages. The languages contain a finite number of predicates of any degree (number of arguments), designating properties of the individuals or relations between them. There are, furthermore, the customary connectives of negation ('$\sim$', corresponding to "not"), disjunction ('$\vee$', "or"), conjunction ('$\cdot$', "and"); universal and existential quantifiers ("for every x," "there is an x"); the sign of identity between individuals '$=$', and 't' as an abbreviation for an arbitrarily chosen tautological sentence. (Thus the languages are certain forms of what is technically known as the lower functional logic with identity.) (The connectives will be used in this paper in three ways, as is customary: (1) between sentences, (2) between predicates (§8), (3) between names (or variables) of sentences (so that, if 'i' and 'j' refer to two sentences, '$i \vee j$' is meant to refer to their disjunction).)

A sentence consisting of a predicate of degree n with n individual constants is called an *atomic sentence* (e.g. 'Pa_1', i.e. 'a_1 has the property P', or 'Ra_3a_5', i.e. 'the relation R holds between a_3 and a_5'). The conjunction of all atomic sentences in a finite language L_N describes one of the possible states of the domain of the N individuals with respect to the properties and relations expressible in the language L_N. If we replace in this conjunction some of the atomic sentences by their negations, we obtain the description of another possible state. All the conjunctions which we can form in this way, including the original one, are called *state-descriptions* in L_N. Analogously, a state-description in L_∞ is a class containing some atomic sentences and the negations of the remaining atomic sentences; since this class is infinite, it cannot be transformed into a conjunction.

In the actual construction of the language systems, which cannot be given here, semantical rules are laid down determining for any given sentence j and any state-description i whether j holds in i, that is to say whether j would be true if i described the actual state among all possible states. The class of those state-descriptions in a language system L (either one of the systems L_N or L_∞) in which j holds is called the *range* of j in L.

The concept of range is fundamental both for deductive and for inductive logic; this has already been pointed out by Wittgenstein. If the range of a

[7] For more detailed explanations of some of these concepts see my *Introduction to Semantics*, 1942.

sentence j in the language system L is universal, i.e. if j holds in every state-description (in L), j must necessarily be true independently of the facts; therefore we call j (in L) in this case *L-true* (logically true, analytic). (The prefix 'L-' stands for "logical"; it is not meant to refer to the system L.) Analogously, if the range of j is null, we call j *L-false* (logically false, self-contradictory). If j is neither L-true nor L-false, we call it *factual* (synthetic, contingent). Suppose that the range of e is included in that of h. Then in every possible case in which e would be true, h would likewise be true. Therefore we say in this case that e *L-implies* (logically implies, entails) h. If two sentences have the same range, we call them *L-equivalent*; in this case, they are merely different formulations for the same content.

The L-concepts just explained are fundamental for deductive logic and therefore also for inductive logic. Inductive logic is constructed out of deductive logic by the introduction of the concept of degree of confirmation. This introduction will here be carried out in three steps: (1) the definition of regular $\mathfrak{c}$-functions (§3), (2) the definition of symmetrical $\mathfrak{c}$-functions (§5), (3) the definition of the degree of confirmation $\mathfrak{c}^*$ (§6).

§3. REGULAR $\mathfrak{c}$-FUNCTIONS

A numerical function $\mathfrak{m}$ ascribing real numbers of the interval 0 to 1 to the sentences of a finite language L_N is called a regular $\mathfrak{m}$-function if it is constructed according to the following rules:

(1) We assign to the state-descriptions in L_N as values of $\mathfrak{m}$ any positive real numbers whose sum is 1.

(2) For every other sentence j in L_N, the value $\mathfrak{m}(j)$ is determined as follows:
  (a) If j is not L-false, $\mathfrak{m}(j)$ is the sum of the $\mathfrak{m}$-values of those state-descriptions which belong to the range of j.
  (b) If j is L-false and hence its range is null, $\mathfrak{m}(j) = 0$.

(The choice of the rule (2)(a) is motivated by the fact that j is L-equivalent to the disjunction of those state-descriptions which belong to the range of j and that these state-descriptions logically exclude each other.)

If any regular $\mathfrak{m}$-function $\mathfrak{m}$ is given, we define a corresponding function $\mathfrak{c}$ as follows:

(3) For any pair of sentences e, h in L_N, where e is not L-false, $\mathfrak{c}(h, e) = \dfrac{\mathfrak{m}(e \cdot h)}{\mathfrak{m}(e)}$.

$\mathfrak{m}(j)$ may be regarded as a measure ascribed to the range of j; thus the function $\mathfrak{m}$ constitutes a metric for the ranges. Since the range of the conjunction $e \cdot h$ is the common part of the ranges of e and of h, the quotient in (3) indicates, so to speak, how large a part of the range of e is included in the range of h. The numerical value of this ratio, however, depends on what particular $\mathfrak{m}$-function has been chosen. We saw earlier that a statement in deductive logic of the form "e L-implies h" says that the range of e is entirely included in that of h. Now we see that a statement in inductive logic of the form "$\mathfrak{c}(h, e) = 3/4$" says that a certain part—in the example, three fourths—of the range of e is included in the range of h.[8] Here, in order to express the partial inclusion numerically, it is necessary to choose a regular $\mathfrak{m}$-function for measuring the ranges. Any $\mathfrak{m}$ chosen leads to a particular $\mathfrak{c}$ as defined above. All functions $\mathfrak{c}$ obtained in this way are called *regular* $\mathfrak{c}$-*functions.*

One might perhaps have the feeling that the metric $\mathfrak{m}$ should not be chosen once for all but should rather be changed according to the accumulating experiences.[9] This feeling is correct in a certain sense. However, it is to be satisfied not by the function $\mathfrak{m}$ used in the definition (3) but by another function $\mathfrak{m}_e$ dependent upon e and leading to an alternative definition (5) for the corresponding $\mathfrak{c}$. If a regular $\mathfrak{m}$ is chosen according to (1) and (2), then a corresponding function $\mathfrak{m}_e$ is defined for the state-descriptions in L_N as follows:

(4) Let i be a state-description in L_N, and e a non-L-false sentence in L_N.
  (a) If e does not hold in i, $\mathfrak{m}_e(i) = 0$.
  (b) If e holds in i, $\mathfrak{m}_e(i) = \dfrac{\mathfrak{m}(i)}{\mathfrak{m}(e)}$.

Thus $\mathfrak{m}_e$ represents a metric for the state-descriptions which changes with the changing evidence e. Now $\mathfrak{m}_e(j)$ for any other sentence j in L_N is defined in analogy to (2) (a) and (b). Then we define the function $\mathfrak{c}$ corresponding to $\mathfrak{m}$ as follows:

(5) For any pair of sentences e, h in L_N, where e is not L-false, $\mathfrak{c}(h, e) = \mathfrak{m}_e(h)$.

It can easily be shown that this alternative definition (5) yields the same values as the original definition (3).

Suppose that a sequence of regular $\mathfrak{m}$-functions is given, one for each of the finite languages L_N ($N = 1, 2$, etc.). Then we define a corresponding $\mathfrak{m}$-function for the infinite language as follows:

(6) $\mathfrak{m}(j)$ in L_∞ is the limit of the values $\mathfrak{m}(j)$ in L_N for $N \rightarrow \infty$.

$\mathfrak{c}$-functions for the finite languages are based on the given $\mathfrak{m}$-functions according to (3). We define a corresponding $\mathfrak{c}$-function for the infinite language as follows:

(7) $\mathfrak{c}(h, e)$ in L_∞ is the limit of the values $\mathfrak{c}(h, e)$ in L_N for $N \rightarrow \infty$.

The definitions (6) and (7) are applicable only in those cases where the specified limits exist.

We shall later see how to select a particular sub-class of regular $\mathfrak{c}$-functions (§5) and finally one particular $\mathfrak{c}$-function $\mathfrak{c}^*$ as the basis of a complete system of inductive logic (§6). For the moment, let us pause at our first step, the definition of regular $\mathfrak{c}$-functions just given, in order to see what results this definition alone can yield, before we add further definitions. The theory of regular $\mathfrak{c}$-functions, i.e. the totality of those theorems which are founded on the definition

[8] See F. Waismann, "Logische Analyse des Wahrscheinlichkeitsbegriffs," *Erkenntnis*, vol. 1, 1930, pp. 228–248.

[9] See Waismann, op. cit., p. 242.

stated, is the first and fundamental part of inductive logic. It turns out that we find here many of the fundamental theorems of the classical theory of probability, e.g. those known as the theorem (or principle) of multiplication, the general and the special theorems of addition, the theorem of division and, based upon it, Bayes' theorem.

One of the cornerstones of the classical theory of probability is the principle of indifference (or principle of insufficient reason). It says that, if our evidence e does not give us any sufficient reason for regarding one of two hypotheses h and h' as more probable than the other, then we must take their probabilities$_1$ as equal: $\mathfrak{c}(h, e) = \mathfrak{c}(h', e)$. Modern authors, especially Keynes, have correctly pointed out that this principle has often been used beyond the limits of its original meaning and has then led to quite absurd results. Moreover, it can easily be shown that, even in its original meaning, the principle is by far too general and leads to contradictions. Therefore the principle must be abandoned. If it is and we consider only those theorems of the classical theory which are provable without the help of this principle, then we find that these theorems hold for all regular $\mathfrak{c}$-functions. The same is true for those modern theories of probability$_1$ (e.g. that by Jeffreys, op.cit.) which make use of the principle of indifference. Most authors of modern axiom systems of probability$_1$ (e.g. Keynes (op.cit.), Waismann (op.cit.), Mazurkiewicz[10], Hosiasson[11], v. Wright[12]) are cautious enough not to accept that principle. An examination of these systems shows that their axioms and hence their theorems hold for all regular $\mathfrak{c}$-functions. Thus these systems restrict themselves to the first part of inductive logic, which, although fundamental and important, constitutes only a very small and weak section of the whole of inductive logic. The weakness of this part shows itself in the fact that it does not determine the value of $\mathfrak{c}$ for any pair h, e except in some special cases where the value is 0 or 1. The theorems of this part tell us merely how to calculate further values of $\mathfrak{c}$ if some values are given. Thus it is clear that this part alone is quite useless for application and must be supplemented by additional rules. (It may be remarked incidentally, that this point marks a fundamental difference between the theories of probability$_1$ and of probability$_2$ which otherwise are analogous in many respects. The theorems concerning probability$_2$ which are analogous to the theorems concerning regular $\mathfrak{c}$-functions constitute not only the first part but the whole of the logico-mathematical theory of probability$_2$. The task of determining the value of probability$_2$ for a given case is—in contradistinction to the corresponding task for probability$_1$—an empirical one and hence lies outside the scope of the logical theory of probability$_2$.)

[10] St. Mazurkiewicz, "Zur Axiomatik der Wahrscheinlichkeitsrechnung," *C. R. Soc. Science Varsovie*, Cl. III, vol. 25, 1932, pp. 1–4.

[11] Janina Hosiasson-Lindenbaum, "On Confirmation," *Journal Symbolic Logic*, vol. 5, 1940, pp. 133–148.

[12] G. H. von Wright, *The Logical Problem of Induction*, (Acta Phil. Fennica, 1941, Fasc. III). See also C. D. Broad, *Mind*, vol. 53, 1944.

§4. THE COMPARATIVE CONCEPT OF CONFIRMATION

Some authors believe that a metrical (or quantitative) concept of degree of confirmation, that is, one with numerical values, can be applied, if at all, only in certain cases of a special kind and that in general we can make only a comparison in terms of higher or lower confirmation without ascribing numerical values. Whether these authors are right or not, the introduction of a merely comparative (or topological) concept of confirmation not presupposing a metrical concept is, in any case, of interest. We shall now discuss a way of defining a concept of this kind.

For technical reasons, we do not take the concept "more confirmed" but "more or equally confirmed." The following discussion refers to the sentences of any finite language L_N. We write, for brevity, "$MC(h, e, h', e')$" for "h is confirmed on the evidence e more highly or just as highly as h' on the evidence e'".

Although the definition of the comparative concept MC at which we aim will not make use of any metrical concept of degree of confirmation, let us now consider, for heuristic purposes, the relation between MC and the metrical concepts, i.e. the regular $\mathfrak{c}$-functions. Suppose we have chosen some concept of degree of confirmation, in other words, a regular $\mathfrak{c}$-function $\mathfrak{c}$, and further a comparative relation MC; then we shall say that MC is in accord with $\mathfrak{c}$ if the following holds:

(1) For any sentences h, e, h', e', if $MC(h, e, h', e')$ then $\mathfrak{c}(h, e) \geqq \mathfrak{c}(h', e')$.

However, we shall not proceed by selecting one $\mathfrak{c}$-function and then choosing a relation MC which is in accord with it. This would not fulfill our intention. Our aim is to find a comparative relation MC which grasps those logical relations between sentences which are, so to speak, prior to the introduction of any particular $\mathfrak{m}$-metric for the ranges and of any particular $\mathfrak{c}$-function; in other words, those logical relations with respect to which all the various regular $\mathfrak{c}$-functions agree. Therefore we lay down the following requirement:

(2) The relation MC is to be defined in such a way that it is in accord with *all* regular $\mathfrak{c}$-functions; in other words, if $MC(h, e, h', e')$, then for every regular $\mathfrak{c}$, $\mathfrak{c}(h, e) \geqq \mathfrak{c}(h', e')$.

It is not difficult to find relations which fulfill this requirement (2). First let us see whether we can find quadruples of sentences h, e, h', e' which satisfy the following condition occurring in (2):

(3) For every regular $\mathfrak{c}$, $\mathfrak{c}(h, e) \geqq \mathfrak{c}(h', e')$.

It is easy to find various kinds of such quadruples. (For instance, if e and e' are any non-L-false sentences, then the condition (3) is satisfied in all cases where e L-implies h, because here $\mathfrak{c}(h, e) = 1$; further in all cases where e' L-implies $\sim h'$, because here $\mathfrak{c}(h', e') = 0$; and in many other cases.) We could, of course, define a relation MC by taking some cases where we know that the condition (3) is satisfied and restricting the relation to these cases. Then the relation would fulfill the requirement (2); however, as long as there are cases which satisfy the condition (3) but which we have not included in the relation, the relation is unnecessarily restricted. Therefore we lay down the following as a second requirement for *MC*:

(4) *MC* is to be defined in such a way that it holds in all cases which satisfy the condition (3); in such a way, in other words, that it is the most comprehensive relation which fulfills the first requirement (2).

These two requirements (2) and (4) together stipulate that $MC(h, e, h', e')$ is to hold if and only if the condition (3) is satisfied; thus the requirements determine uniquely one relation *MC*. However, because they refer to the c-functions, we do not take these requirements as a definition for *MC*, for we intend to give a purely comparative definition for *MC*, a definition which does not make use of any metrical concepts but which leads nevertheless to a relation *MC* which fulfills the requirements (2) and (4) referring to c-functions. This aim is reached by the following definition (where '$=_{Df}$' is used as sign of definition).

(5) $MC(h, e, h', e') =_{Df}$ the sentences h, e, h', e' (in L_N) are such that e and e' are not L-false and at least one of the following three conditions is fulfilled:
(a) e L-implies h,
(b) e' L-implies $\sim h'$,
(c) $e' \cdot h'$ L-implies $e \cdot h$ and simultaneously e L-implies $h \vee e'$.

((a) and (b) are the two kinds of rather trivial cases earlier mentioned; (c) comprehends the interesting cases; an explanation and discussion of them cannot be given here.)

The following theorem can then be proved concerning the relation *MC* defined by (5). It shows that this relation fulfills the two requirements (2) and (4).

(6) For any sentences h, e, h', e' in L_N the following holds:
(a) If $MC(h, e, h', e')$, then, for every regular $\mathfrak{c}$, $\mathfrak{c}(h, e) \geqq \mathfrak{c}(h', e')$.
(b) If, for every regular $\mathfrak{c}$, $\mathfrak{c}(h, e) \geqq \mathfrak{c}(h', e')$, then $MC(h, e, h', e')$.

(With respect to L_∞, the analogue of (6)(a) holds for all sentences, and that of (6)(b) for all sentences without variables.)

§5. SYMMETRICAL c-FUNCTIONS

The next step in the construction of our system of inductive logic consists in selecting a narrow sub-class of the comprehensive class of all regular c-functions. The guiding idea for this step will be the principle that inductive logic should treat all individuals on a par. The same principle holds for deductive logic; for instance, if '$\cdot\cdot a \cdot\cdot b \cdot\cdot$' L-implies '--$b$--$c$--' (where the first expression in quotation marks is meant to indicate some sentence containing 'a' and 'b', and the second another sentence containing 'b' and 'c'), then L-implication holds likewise between corresponding sentences with other individual constants, e.g. between '$\cdot\cdot d \cdot\cdot c \cdot\cdot$' and '--$c$--$a$--'. Now we require that this should hold also for inductive logic, e.g. that $\mathfrak{c}$('--b--c--', '$\cdot\cdot a \cdot\cdot b \cdot\cdot$') = $\mathfrak{c}$('--c--a--', '$\cdot\cdot d \cdot\cdot c \cdot\cdot$'). It seems that all authors on probability$_1$ have assumed this principle—although it has seldom, if ever, been stated explicitly—by formulating theorems in the following or similar terms: "On the basis of observations of s things of which s_1 were found to have the property M and s_2 not to have this property, the probability that another thing has this property is such and such." The fact that these theorems refer only to the number of things observed and do not mention particular things shows implicitly that it does not matter which things are involved; thus it is assumed, e.g., that $\mathfrak{c}(\text{'}Pd\text{'}, \text{'}Pa \cdot Pb \cdot {\sim}Pc\text{'}) = \mathfrak{c}(\text{'}Pc\text{'}, \text{'}Pa \cdot Pd \cdot {\sim}Pb\text{'})$.

The principle could also be formulated as follows. Inductive logic should, like deductive logic, make no discrimination among individuals. In other words, the value of $\mathfrak{c}$ should be influenced only by those differences between individuals which are expressed in the two sentences involved; no differences between particular individuals should be stipulated by the rules of either deductive or inductive logic.

It can be shown that this principle of non-discrimination is fulfilled if $\mathfrak{c}$ belongs to the class of symmetrical $\mathfrak{c}$-functions which will now be defined. Two state-descriptions in a language L_N are said to be *isomorphic* or to have the same structure if one is formed from the other by replacements of the following kind: we take any one-one relation R such that both its domain and its converse domain is the class of all individual constants in L_N, and then replace every individual constant in the given state-description by the one correlated with it by R. If a regular $\mathfrak{m}$-function (for L_N) assigns to any two isomorphic state-descriptions (in L_N) equal values, it is called a symmetrical $\mathfrak{m}$-function; and a $\mathfrak{c}$-function based upon such an $\mathfrak{m}$-function in the way explained earlier (see (3) in §3) is then called a *symmetrical* $\mathfrak{c}$*-function.*

§6. THE DEGREE OF CONFIRMATION $\mathfrak{c}^*$

Let i be a state-description in L_N. Suppose there are n_i state-descriptions in L_N isomorphic to i (including i itself), say i, i', i'', etc. These n_i state-descriptions exhibit one and the same structure of the universe of L_N with respect to all the properties and relations designated by the primitive predicates in L_N. This concept of structure is an extension of the concept of structure or relation-number (Russell) usually applied to one dyadic relation. The common structure of the isomorphic state-descriptions i, i', i'', etc. can be described by their disjunction $i \vee i' \vee i'' \vee \cdots$. Therefore we call this disjunction, say j, a *structure-description* in L_N. It can be shown that the range of j contains only the isomorphic state-descriptions i, i', i'', etc. Therefore (see (2)(a) in §3) $\mathfrak{m}(j)$ is the sum of the $\mathfrak{m}$-values for these state-descriptions. If $\mathfrak{m}$ is symmetrical, then these values are equal, and hence

(1) $\mathfrak{m}(j) = n_i \times \mathfrak{m}(i).$

And, conversely, if $\mathfrak{m}(j)$ is known to be q, then

(2) $\mathfrak{m}(i) = \mathfrak{m}(i') = \mathfrak{m}(i'') = \cdots = q/n_i.$

This shows that what remains to be decided, is merely the distribution of $\mathfrak{m}$-values among the structure-descriptions in L_N. We decide to give them equal $\mathfrak{m}$-values. This decision constitutes the third step in the construction of our inductive logic. This step leads to one particular $\mathfrak{m}$-function $\mathfrak{m}^*$ and to the $\mathfrak{c}$-function $\mathfrak{c}^*$ based upon $\mathfrak{m}^*$. According to the preceding discussion, $\mathfrak{m}^*$ is characterized by the following two stipulations:

(3) (a) $\mathfrak{m}^*$ is a symmetrical $\mathfrak{m}$-function;
(b) $\mathfrak{m}^*$ has the same value for all structure-descriptions (in L_N).

We shall see that these two stipulations characterize just one function. Every state-description (in L_N) belongs to the range of just one structure-description. Therefore, the sum of the $\mathfrak{m}^*$-values for all structure-descriptions in L_N must be the same as for all state-descriptions, hence 1 (according to (1) in §3). Thus, if the number of structure-descriptions in L_N is m, then, according to (3)(b),

(4) for every structure-description j in L_N, $\mathfrak{m}^*(j) = \dfrac{1}{m}$.

Therefore, if i is any state-description in L_N and n_i is the number of state-descrip tions isomorphic to i, then, according to (3)(a) and (2),

$$\text{(5)} \qquad \mathfrak{m}^*(i) = \frac{1}{mn_i}.$$

(5) constitutes a definition of $\mathfrak{m}^*$ as applied to the state-descriptions in L_N. On this basis, further definitions are laid down as explained above (see (2) and (3) in §3): first a definition of $\mathfrak{m}^*$ as applied to all sentences in L_N, and then a definition of $\mathfrak{c}^*$ on the basis of $\mathfrak{m}^*$. Our inductive logic is the theory of this particular function $\mathfrak{c}^*$ as our concept of degree of confirmation.

It seems to me that there are good and even compelling reasons for the stipulation (3)(a), i.e. the choice of a symmetrical function. The proposal of any non-symmetrical $\mathfrak{c}$-function as degree of confirmation could hardly be regarded as acceptable. The same can not be said, however, for the stipulation (3)(b). No doubt, to the way of thinking which was customary in the classical period of the theory of probability, (3)(b) would appear as validated, like (3)(a), by the principle of indifference. However, to modern, more critical thought, this mode of reasoning appears as invalid because the structure-descriptions (in contradistinction to the individual constants) are by no means alike in their logical features but show very conspicuous differences. The definition of $\mathfrak{c}^*$ shows a great simplicity in comparison with other concepts which may be taken into consideration. Although this fact may influence our decision to choose $\mathfrak{c}^*$, it cannot, of course, be regarded as a sufficient reason for this choice. It seems to me that the choice of $\mathfrak{c}^*$ cannot be justified by any features of the definition which are immediately recognizable, but only by the consequences to which the definition leads.

There is another $\mathfrak{c}$-function $\mathfrak{c}_W$ which at the first glance appears not less plausible than $\mathfrak{c}^*$. The choice of this function may be suggested by the following consideration. Prior to experience, there seems to be no reason to regard one state-description as less probable than another. Accordingly, it might seem natural to assign equal $\mathfrak{m}$-values to all state-descriptions. Hence, if the number of the state-descriptions in L_N is n, we define for any state-description i

$$\mathfrak{m}_W(i) = 1/n. \tag{6}$$

This definition (6) for $\mathfrak{m}_W$ is even simpler than the definition (5) for $\mathfrak{m}^*$. The measure ascribed to the ranges is here simply taken as proportional to the cardinal numbers of the ranges. On the basis of the $\mathfrak{m}_W$-values for the state-descriptions defined by (6), the values for the sentences are determined as before (see (2) in §3), and then $\mathfrak{c}_W$ is defined on the basis of $\mathfrak{m}_W$ (see (3) in §3).[13]

In spite of its apparent plausibility, the function $\mathfrak{c}_W$ can easily be seen to be entirely inadequate as a concept of degree of confirmation. As an example, consider the language L_{101} with 'P' as the only primitive predicate. Let the number of state-descriptions in this language be n (it is 2^{101}). Then for any state-description, $\mathfrak{m}_W = 1/n$. Let e be the conjunction $Pa_1 \cdot Pa_2 \cdot Pa_3 \cdots Pa_{100}$ and let h be 'Pa_{101}'. Then $e \cdot h$ is a state-description and hence $\mathfrak{m}_W(e \cdot h) = 1/n$. e holds only in the two state-descriptions $e \cdot h$ and $e \cdot \sim h$; hence $\mathfrak{m}_W(e) = 2/n$. Therefore $\mathfrak{c}_W(h, e) = \frac{1}{2}$. If e' is formed from e by replacing some or even all of the atomic sentences with their negations, we obtain likewise $\mathfrak{c}_W(h, e') = \frac{1}{2}$. Thus the $\mathfrak{c}_W$-value for the prediction that a_{101} is P is always the same, no matter whether among the hundred observed individuals the number of those which we have found to be P is 100 or 50 or 0 or any other number. Thus the choice of $\mathfrak{c}_W$ as the degree of confirmation would be tantamount to the principle never to let our past experiences influence our expectations for the future. This would obviously be in striking contradiction to the basic principle of all inductive reasoning.

§7. LANGUAGES WITH ONE-PLACE PREDICATES ONLY

The discussions in the rest of this paper concern only those language systems whose primitive predicates are one-place predicates and hence designate properties, not relations. It seems that all theories of probability constructed so far have restricted themselves, or at least all of their important theorems, to properties. Although the definition of $\mathfrak{c}^*$ in the preceding section has been stated in a

[13] It seems that Wittgenstein meant this function $\mathfrak{c}_W$ in his definition of probability, which he indicates briefly without examining its consequences. In his *Tractatus Logico-Philosophicus*, he says: "A proposition is the expression of agreement and disagreement with the truth-possibilities of the elementary [i.e. atomic] propositions" (*4.4); "The world is completely described by the specification of all elementary propositions plus the specification, which of them are true and which false" (*4.26). The truth-possibilities specified in this way correspond to our state-descriptions. Those truth-possibilities which verify a given proposition (in our terminology, those state-descriptions in which a given sentence holds) are called the truth-grounds of that proposition (*5.101). "If T_r is the number of the truth-grounds of the proposition "r", T_{rs} the number of those truth-grounds of the proposition "s" which are at the same time truth-grounds of "r", then we call the ratio $T_{rs}:T_r$ the measure of the *probability* which the proposition "r" gives to the proposition "s" "(*5.15). It seems that the concept of probability thus defined coincides with the function $\mathfrak{c}_W$.

general way so as to apply also to languages with relations, the greater part of our inductive logic will be restricted to properties. An extension of this part of inductive logic to relations would require certain results in the deductive logic of relations, results which this discipline, although widely developed in other respects, has not yet reached (e.g. an answer to the apparently simple question as to the number of structures in a given finite language system).

Let L_N^p be a language containing N individual constants 'a_1', $\cdots$ 'a_N', and p one-place primitive predicates 'P_1', $\cdots$ 'P_p'. Let us consider the following expressions (sentential matrices). We start with '$P_1x \cdot P_2x \cdots P_px$'; from this expression we form others by negating some of the conjunctive components, until we come to '$\sim P_1x \cdot \sim P_2x \cdots \sim P_px$', where all components are negated. The number of these expressions is $k = 2^p$; we abbreviate them by 'Q_1x', $\cdots$ 'Q_kx'. We call the k properties expressed by those k expressions in conjunctive form and now designated by the k new Q-predicates the *Q-properties* with respect to the given language L_N^p. We see easily that these Q-properties are the strongest properties expressible in this language (except for the L-empty, i.e., logically self-contradictory, property); and further, that they constitute an exhaustive and non-overlapping classification, that is to say, every individual has one and only one of the Q-properties. Thus, if we state for each individual which of the Q-properties it has, then we have described the individuals completely. Every state-description can be brought into the form of such a statement, i.e. a conjunction of N Q-sentences, one for each of the N individuals. Suppose that in a given state-description i the number of individuals having the property Q_1 is N_1, the number for Q_2 is N_2, $\cdots$ that for Q_k is N_k. Then we call the numbers N_1, N_2, $\cdots$ N_k the *Q-numbers* of the state-description i; their sum is N. Two state-descriptions are isomorphic if and only if they have the same Q-numbers. Thus here a structure-description is a statistical description giving the Q-numbers N_1, N_2, etc., without specifying which individuals have the properties Q_1, Q_2, etc.

Here—in contradistinction to languages with relations—it is easy to find an explicit function for the number m of structure-descriptions and, for any given state-description i with the Q-numbers N_1, $\cdots$ N_k, an explicit function for the number n_i of state-descriptions isomorphic to i, and hence also a function for $\mathfrak{m}^*(i)$.[14]

Let j be a non-general sentence (i.e. one without variables) in L_N^p. Since

[14] The results are as follows.

(1) $$m = \frac{(N + k - 1)!}{N!(k - 1)!}$$

(2) $$n_i = \frac{N!}{N_1!\, N_2! \cdots N_k!}$$

Therefore (according to (5) in §6):

(3) $$\mathfrak{m}^*(i) = \frac{N_1!\, N_2! \cdots N_k!(k - 1)!}{(N + k - 1)!}$$

there are effective procedures (that is, sets of fixed rules furnishing results in a finite number of steps) for constructing all state-descriptions in which j holds and for computing $\mathfrak{m}^*$ for any given state-description, these procedures together yield an effective procedure for computing $\mathfrak{m}^*(j)$ (according to (2) in §3). However, the number of state-descriptions becomes very large even for small language systems (it is k^N, hence, e.g., in L_7^3 it is more than two million.) Therefore, while the procedure indicated for the computation of $\mathfrak{m}^*(j)$ is effective, nevertheless in most ordinary cases it is impracticable; that is to say, the number of steps to be taken, although finite, is so large that nobody will have the time to carry them out to the end. I have developed another procedure for the computation of $\mathfrak{m}^*(j)$ which is not only effective but also practicable if the number of individual constants occurring in j is not too large.

The value of $\mathfrak{m}^*$ for a sentence j in the infinite language has been defined (see (6) in §3) as the limit of its values for the same sentence j in the finite languages. The question arises whether and under what conditions this limit exists. Here we have to distinguish two cases. (i) Suppose that j contains no variable. Here the situation is simple; it can be shown that in this case $\mathfrak{m}^*(j)$ is the same in all finite languages in which j occurs; hence it has the same value also in the infinite language. (ii) Let j be general, i.e., contain variables. Here the situation is quite different. For a given finite language with N individuals, j can of course easily be transformed into an L-equivalent sentence j'_N without variables, because in this language a universal sentence is L-equivalent to a conjunction of N components. The values of $\mathfrak{m}^*(j'_N)$ are in general different for each N; and although the simplified procedure mentioned above is available for the computation of these values, this procedure becomes impracticable even for moderate N. Thus for general sentences the problem of the existence and the practical computability of the limit becomes serious. It can be shown that for every general sentence the limit exists; hence $\mathfrak{m}^*$ has a value for all sentences in the infinite language. Moreover, an effective procedure for the computation of $\mathfrak{m}^*(j)$ for any sentence j in the infinite language has been constructed. This is based on a procedure for transforming any given general sentence j into a non-general sentence j' such that j and j', although not necessarily L-equivalent, have the same $\mathfrak{m}^*$-value in the infinite language and j' does not contain more individual constants than j; this procedure is not only effective but also practicable for sentences of customary length. Thus, the computation of $\mathfrak{m}^*(j)$ for a general sentence j is in fact much simpler for the infinite language than for a finite language with a large N.

With the help of the procedure mentioned, the following theorem is obtained:

If j is a purely general sentence (i.e. one without individual constants) in the infinite language, then $\mathfrak{m}^*(j)$ is either 0 or 1.

§8. INDUCTIVE INFERENCES

One of the chief tasks of inductive logic is to furnish general theorems concerning inductive inferences. We keep the traditional term "inference"; however, we do not mean by it merely a transition from one sentence to another (viz.

from the evidence or premiss e to the hypothesis or conclusion h) but the determination of the degree of confirmation $\mathfrak{c}(h, e)$. In deductive logic it is sufficient to state that h follows with necessity from e; in inductive logic, on the other hand, it would not be sufficient to state that h follows—not with necessity but to some degree or other—from e. It must be specified to what degree h follows from e; in other words, the value of $\mathfrak{c}(h, e)$ must be given. We shall now indicate some results with respect to the most important kinds of inductive inference. These inferences are of special importance when the evidence or the hypothesis or both give statistical information, e.g. concerning the absolute or relative frequencies of given properties.

If a property can be expressed by primitive predicates together with the ordinary connectives of negation, disjunction, and conjunction (without the use of individual constants, quantifiers, or the identity sign), it is called an *elementary property*. We shall use 'M', 'M'', 'M_1', 'M_2', etc. for elementary properties. If a property is empty by logical necessity (e.g. the property designated by '$P \cdot \sim P$') we call it L-empty; if it is universal by logical necessity (e.g. '$P \vee \sim P$'), we call it L-universal. If it is neither L-empty nor L-universal (e.g. 'P_1', '$P_1 \cdot \sim P_2$'), we call it a *factual property*; in this case it may still happen to be universal or empty, but if so, then contingently, not necessarily. It can be shown that every elementary property which is not L-empty is uniquely analysable into a disjunction (i.e. or-connection) of Q-properties. If M is a disjunction of n Q-properties ($n \geqq 1$), we say that the (logical) *width* of M is n; to an L-empty property we ascribe the width 0. If the width of M is w ($\geqq 0$), we call w/k its *relative width* (k is the number of Q-properties).

The concepts of width and relative width are very important for inductive logic. Their neglect seems to me one of the decisive defects in the classical theory of probability which formulates its theorems "for any property" without qualification. For instance, Laplace takes the probability a priori that a given thing has a given property, no matter of what kind, to be $\frac{1}{2}$. However, it seems clear that this probability cannot be the same for a very strong property (e.g. '$P_1 \cdot P_2 \cdot P_3$') and for a very weak property (e.g. '$P_1 \vee P_2 \vee P_3$'). According to our definition, the first of the two properties just mentioned has the relative width $\frac{1}{8}$, and the second $\frac{7}{8}$. In this and in many other cases the probability or degree of confirmation must depend upon the widths of the properties involved. This will be seen in some of the theorems to be mentioned later.

§9. THE DIRECT INFERENCE

Inductive inferences often concern a situation where we investigate a whole population (of persons, things, atoms, or whatever else) and one or several samples picked out of the population. An inductive inference from the whole population to a sample is called a direct inductive inference. For the sake of simplicity, we shall discuss here and in most of the subsequent sections only the case of one property M, hence a classification of all individuals into M and $\sim M$. The theorems for classifications with more properties are analogous but more complicated. In the present case, the evidence e says that in a whole population of n individuals there are n_1 with the property M and $n_2 = n - n_1$ with $\sim M$; hence the relative frequency of M is $r = n_1/n$. The hypothesis h says that a sample of s individuals taken from the whole population will contain s_1 individuals with the property M and $s_2 = s - s_1$ with $\sim M$. Our theory yields in this case the same values as the classical theory.[15]

If we vary s_1, then $\mathfrak{c}^*$ has its maximum in the case where the relative frequency s_1/s in the sample is equal or close to that in the whole population.

If the sample consists of only one individual c, and h says that c is M, then $\mathfrak{c}^*(h, e) = r$.

As an approximation in the case that n is very large in relation to s, Newton's theorem holds.[16] If furthermore the sample is sufficiently large, we obtain as an approximation Bernoulli's theorem in its various forms.

It is worthwhile to note two characteristics which distinguish the direct inductive inference from the other inductive inferences and make it, in a sense, more closely related to deductive inferences:

(i) The results just mentioned hold not only for $\mathfrak{c}^*$ but likewise for all symmetrical $\mathfrak{c}$-functions; in other words, the results are independent of the particular $\mathfrak{m}$-metric chosen provided only that it takes all individuals on a par.

(ii) The results are independent of the width of M. This is the reason for the agreement between our theory and the classical theory at this point.

§10. THE PREDICTIVE INFERENCE

We call the inference from one sample to another the predictive inference. In this case, the evidence e says that in a first sample of s individuals, there are s_1 with the property M, and $s_2 = s - s_1$ with $\sim M$. The hypothesis h says that in a second sample of s' other individuals, there will be s_1' with M, and $s_2' = s' - s_1'$ with $\sim M$. Let the width of M be w_1; hence the width of $\sim M$ is $w_2 = k - w_1$.[K]

[15] The general theorem is as follows:

$$\mathfrak{c}^*(h, e) = \frac{\binom{n_1}{s_1}\binom{n_2}{s_1}}{\binom{n}{s}}.$$

[16]
$$\mathfrak{c}^*(h, e) = \binom{s}{s_1} r^{s_1}(1 - r)^{s_2}.$$

[17] The general theorem is as follows:

$$\mathfrak{c}^*(h, e) = \frac{\binom{s_1 + s_1' + w_1 - 1}{s_1'}\binom{s_2 + s_2' + w_2 - 1}{s_2'}}{\binom{s + s' + k - 1}{s'}}.$$

The most important special case is that where h refers to one individual c only and says that c is M. In this case,

$$\text{(1)} \qquad \mathfrak{c}^*(h, e) = \frac{s_1 + w_1}{s + k}.$$

Laplace's much debated rule of succession gives in this case simply the value $\frac{s_1 + 1}{s + 2}$ for any property whatever; this, however, if applied to different properties, leads to contradictions. Other authors state the value s_1/s, that is, they take simply the observed relative frequency as the probability for the prediction that an unobserved individual has the property in question. This rule, however, leads to quite implausible results. If $s_1 = s$, e.g., if three individuals have been observed and all of them have been found to be M, the last-mentioned rule gives the probability for the next individual being M as 1, which seems hardly acceptable. According to (1), $\mathfrak{c}^*$ is influenced by the following two factors (though not uniquely determined by them):

(i) w_1/k, the relative width of M;
(ii) s_1/s, the relative frequency of M in the observed sample.

The factor (i) is purely logical; it is determined by the semantical rules. (ii) is empirical; it is determined by observing and counting the individuals in the sample. The value of $\mathfrak{c}^*$ always lies between those of (i) and (ii). Before any individual has been observed, $\mathfrak{c}^*$ is equal to the logical factor (i). As we first begin to observe a sample, $\mathfrak{c}^*$ is influenced more by this factor than by (ii). As the sample is increased by observing more and more individuals (but not including the one mentioned in h), the empirical factor (ii) gains more and more influence upon $\mathfrak{c}^*$ which approaches closer and closer to (ii); and when the sample is sufficiently large, $\mathfrak{c}^*$ is practically equal to the relative frequency (ii). These results seem quite plausible.[13]

The predictive inference is the most important inductive inference. The kinds of inference discussed in the subsequent sections may be construed as special cases of the predictive inference.

[13] Another theorem may be mentioned which deals with the case where, in distinction to the case just discussed, the evidence already gives some information about the individual c mentioned in h. Let M_1 be a factual elementary property with the width w_1 ($w_1 \geqq 2$); thus M_1 is a disjunction of w_1 Q-properties. Let M_2 be the disjunction of w_2 among those w_1 Q-properties ($1 \leqq w_2 < w_1$); hence M_2 L-implies M_1 and has the width w_2. e specifies first how the s individuals of an observed sample are distributed among certain properties, and, in particular, it says that s_1 of them have the property M_1 and s_2 of these s_1 individuals have also the property M_2; in addition, e says that c is M_1; and h says that c is also M_2. Then,

$$\mathfrak{c}^*(h, e) = \frac{s_2 + w_2}{s_1 + w_1}.$$

This is analogous to (1); but in the place of the whole sample we have here that part of it which shows the property M_1.

§11. THE INFERENCE BY ANALOGY

The inference by analogy applies to the following situation. The evidence known to us is the fact that individuals b and c agree in certain properties and, in addition, that b has a further property; thereupon we consider the hypothesis that c too has this property. Logicians have always felt that a peculiar difficulty is here involved. It seems plausible to assume that the probability of the hypothesis is the higher the more properties b and c are known to have in common; on the other hand, it is felt that these common properties should not simply be counted but weighed in some way. This becomes possible with the help of the concept of width. Let M_1 be the conjunction of all properties which b and c are known to have in common. The known similarity between b and c is the greater the stronger the property M_1, hence the smaller its width. Let M_2 be the conjunction of all properties which b is known to have. Let the width of M_1 be w_1, and that of M_2, w_2. According to the above description of the situation, we presuppose that M_2 L-implies M_1 but is not L-equivalent to M_1; hence $w_1 > w_2$. Now we take as evidence the conjunction $e \cdot j$; e says that b is M_2, and j says that c is M_1. The hypothesis h says that c has not only the properties ascribed to it in the evidence but also the one (or several) ascribed in the evidence to b only, in other words, that c has all known properties of b, or briefly that c is M_2. Then

$$(1) \qquad \mathfrak{c}^*(h, e \cdot j) = \frac{w_2 + 1}{w_1 + 1}.$$

j and h speak only about c; e introduces the other individual b which serves to connect the known properties of c expressed by j with its unknown properties expressed by h. The chief question is whether the degree of confirmation of h is increased by the analogy between c and b, in other words, by the addition of e to our knowledge. A theorem[19] is found which gives an affirmative answer to this question. However, the increase of $\mathfrak{c}^*$ is under ordinary conditions rather small; this is in agreement with the general conception according to which reasoning by analogy, although admissible, can usually yield only rather weak results.

Hosiasson[20] has raised the question mentioned above and discussed it in detail. She says that an affirmative answer, a proof for the increase of the degree of confirmation in the situation described, would justify the universally accepted reasoning by analogy. However, she finally admits that she does not find such a proof on the basis of her axioms. I think it is not astonishing that neither the classical theory nor modern theories of probability have been able to give a satisfactory account of and justification for the inference by analogy. For, as the theorems mentioned show, the degree of confirmation and its increase depend

19
$$\frac{\mathfrak{c}^*(h, e \cdot j)}{\mathfrak{c}^*(h, j)} = 1 + \frac{w_1 - w_2}{w_2(w_1 + 1)}$$

This theorem shows that the ratio of the increase of $\mathfrak{c}^*$ is greater than 1, since $w_1 > w_2$.

20 Janina Lindenbaum-Hosiasson, "Induction et analogie: Comparaison de leur fondement," *Mind*, vol. 50, 1941, pp. 351–365; see especially pp. 361–365.

here not on relative frequencies but entirely on the logical widths of the properties involved, thus on magnitudes neglected by both classical and modern theories.

The case discussed above is that of simple analogy. For the case of multiple analogy, based on the similarity of c not only with one other individual but with a number n of them, similar theorems hold. They show that $\mathfrak{c}^*$ increases with increasing n and approaches 1 asymptotically. Thus, multiple analogy is shown to be much more effective than simple analogy, as seems plausible.

§12. THE INVERSE INFERENCE

The inference from a sample to the whole population is called the inverse inductive inference. This inference can be regarded as a special case of the predictive inference with the second sample covering the whole remainder of the population. This inference is of much greater importance for practical statistical work than the direct inference, because we usually have statistical information only for some samples and not for the whole population.

Let the evidence e say that in an observed sample of s individuals there are s_1 individuals with the property M and $s_2 = s - s_1$ with $\sim M$. The hypothesis h says that in the whole population of n individuals, of which the sample is a part, there are n_1 individuals with M and n_2 with $\sim M$ ($n_1 \geqq s_1$, $n_2 \geqq s_2$). Let the width of M be w_1, and that of $\sim M$ be $w_2 = k - w_1$. Here, in distinction to the direct inference, $\mathfrak{c}^*(h, e)$ is dependent not only upon the frequencies but also upon the widths of the two properties.[21]

§13. THE UNIVERSAL INFERENCE

The universal inductive inference is the inference from a report on an observed sample to a hypothesis of universal form. Sometimes the term 'induction' has been applied to this kind of inference alone, while we use it in a much wider sense for all non-deductive kinds of inference. The universal inference is not even the most important one; it seems to me now that the role of universal sentences in the inductive procedures of science has generally been overestimated. This will be explained in the next section.

Let us consider a simple law l, i.e. a factual universal sentence of the form "all M are M'" or, more exactly, "for every x, if x is M, then x is M'", where M and M' are elementary properties. As an example, take "all swans are white". Let us abbreviate '$M \cdot \sim M'$' ("non-white swan") by 'M_1' and let the width of

[21] The general theorem is as follows:

$$\mathfrak{c}^*(h, e) = \frac{\binom{n_1 + w_1 - 1}{s_1 + w_1 - 1}\binom{n_2 + w_2 - 1}{s_2 + w_2 - 1}}{\binom{n + k - 1}{n - s}}.$$

Other theorems, which cannot be stated here, concern the case where more than two properties are involved, or give approximations for the frequent case where the whole population is very large in relation to the sample.

M_1 be w_1. Then l can be formulated thus: "M_1 is empty", i.e. "there is no individual (in the domain of individuals of the language in question) with the property M_1" ("there are no non-white swans"). Since l is a factual sentence, M_1 is a factual property; hence $w_1 > 0$. To take an example, let w_1 be 3; hence M_1 is a disjunction of three Q-properties, say $Q \vee Q' \vee Q''$. Therefore, l can be transformed into: "Q is empty, and Q' is empty, and Q'' is empty". The weakest factual laws in a language are those which say that a certain Q-property is empty; we call them Q-laws. Thus we see that l can be transformed into a conjunction of w_1 Q-laws. Obviously l asserts more if w_1 is larger; therefore we say that the law l has the strength w_1.

Let the evidence e be a report about an observed sample of s individuals such that we see from e that none of these s individuals violates the law l; that is to say, e ascribes to each of the s individuals either simply the property $\sim M_1$ or some other property L-implying $\sim M_1$. Let l, as above, be a simple law which says that M_1 is empty, and w_1 be the width of M_1; hence the width of $\sim M_1$ is $w_2 = k - w_1$. For finite languages with N individuals, $c^*(l, e)$ is found to decrease with increasing N, as seems plausible.[22] If N is very large, c^* becomes very small; and for an infinite universe it becomes 0. The latter result may seem astonishing at first sight; it seems not in accordance with the fact that scientists often speak of "well-confirmed" laws. The problem involved here will be discussed later.

So far we have considered the case in which only positive instances of the law l have been observed. Inductive logic must, however, deal also with the case of negative instances. Therefore let us now examine another evidence e' which says that in the observed sample of s individuals there are s_1 which have the property M_1 (non-white swans) and hence violate the law l, and that $s_2 = s - s_1$ have $\sim M_1$ and hence satisfy the law l. Obviously, in this case there is no point in taking as hypothesis the law l in its original forms, because l is logically incom-

[22] The general theorem is as follows:

$$(1) \qquad c^*(l, e) = \frac{\binom{s + k - 1}{w_1}}{\binom{N + k - 1}{w_1}}.$$

In the special case of a language containing 'M_1' as the only primitive predicate, we have $w_1 = 1$ and $k = 2$, and hence $c^*(l, e) = \frac{s+1}{N+1}$. The latter value is given by some authors as holding generally (see Jeffreys, op.cit., p. 106 (16)). However, it seems plausible that the degree of confirmation must be smaller for a stronger law and hence depend upon w_1.

If s, and hence N, too, is very large in relation to k, the following holds as an approximation:

$$(2) \qquad c^*(l, e) = \left(\frac{s}{N}\right)^{w_1}.$$

For the infinite language L_∞ we obtain, according to definition (7) in §3:

$$(3) \qquad c^*(l, e) = 0.$$

patible with the present evidence e', and hence $\mathfrak{c}^*(l, e') = 0$. That all individuals satisfy l is excluded by e'; the question remains whether at least all unobserved individuals satisfy l. Therefore we take here as hypothesis the restricted law l' corresponding to the original unrestricted law l; l' says that all individuals not belonging to the sample of s individuals described in e' have the property $\sim M_1$. w_1 and w_2 are, as previously, the widths of M_1 and $\sim M_1$ respectively. It is found that $\mathfrak{c}^*(l', e')$ decreases with an increase of N and even more with an increase in the number s_1 of violating cases.[23] It can be shown that, under ordinary circumstances with large N, $\mathfrak{c}^*$ increases moderately when a new individual is observed which satisfies the original law l. On the other hand, if the new individual violates l, $\mathfrak{c}^*$ decreases very much, its value becoming a small fraction of its previous value. This seems in good agreement with the general conception.

For the infinite universe, $\mathfrak{c}^*$ is again 0, as in the previous case. This result will be discussed in the next section.

§14. THE INSTANCE CONFIRMATION OF A LAW

Suppose we ask an engineer who is building a bridge why he has chosen the building materials he is using, the arrangement and dimensions of the supports, etc. He will refer to certain physical laws, among them some general laws of mechanics and some specific laws concerning the strength of the materials. On further inquiry as to his confidence in these laws he may apply to them phrases like "very reliable", "well founded", "amply confirmed by numerous experiences". What do these phrases mean? It is clear that they are intended to say something about probability$_1$ or degree of confirmation. Hence, what is meant could be formulated more explicitly in a statement of the form "$\mathfrak{c}(h, e)$ is high" or the like. Here the evidence e is obviously the relevant observational knowledge of the engineer or of all physicists together at the present time. But what is to serve as the hypothesis h? One might perhaps think at first that h is the law in question, hence a universal sentence l of the form: "For every space-time point x, if such and such conditions are fulfilled at x, then such and such is the case at x". I think, however, that the engineer is chiefly interested not in this sentence l, which speaks about an immense number, perhaps an infinite number, of instances dispersed through all time and space, but rather in one instance of l or a relatively small number of instances. When he says that the law is very reliable, he does not mean to say that he is willing to bet that among the billion of billions, or an infinite number, of instances to which the law applies there is not one counter-instance, but merely that this bridge will not be a counter-instance, or that among all bridges which he will construct during his lifetime, or among those which all engineers will construct during the next one

[23] The theorem is as follows:

$$\mathfrak{c}^*(l', e') = \frac{\binom{s + k - 1}{s_1 + w_1}}{\binom{N + k - 1}{s_1 + w_1}}.$$

thousand years, there will be no counter-instance. Thus h is not the law l itself but only a prediction concerning one instance or a relatively small number of instances. Therefore, what is vaguely called the reliability of a law is measured not by the degree of confirmation of the law itself but by that of one or several instances. This suggests the subsequent definitions. They refer, for the sake of simplicity, to just one instance; the case of several, say one hundred, instances can then easily be judged likewise. Let e be any non-L-false sentence without variables. Let l be a simple law of the form earlier described (§13). Then we understand by the *instance confirmation* of l on the evidence e, in symbols "$c_i^*(l, e)$", the degree of confirmation, on the evidence e, of the hypothesis that a new individual not mentioned in e fulfills the law l.[24]

The second concept, now to be defined, seems in many cases to represent still more accurately what is vaguely meant by the reliability of a law l. We suppose here that l has the frequently used conditional form mentioned earlier: "For every x, if x is M, then x is M'" (e.g. "all swans are white"). By the *qualified-instance confirmation* of the law that all swans are white we mean the degree of confirmation for the hypothesis h' that the next swan to be observed will likewise be white. The difference between the hypothesis h used previously for the instance confirmation and the hypothesis h' just described consists in the fact that the latter concerns an individual which is already qualified as fulfilling the condition M. That is the reason why we speak here of the qualified-instance confirmation, in symbols "c_{qi}^*".[25] The results obtained concerning instance confirmation and qualified-instance confirmation[26] show that the values of these two functions are independent of N and hence hold for all finite and infinite universes. It has been found that, if the number s_1 of observed counter-instances

[24] In technical terms, the definition is as follows:
$c_i^*(l, e) =_{Df} c^*(h, e)$, where h is an instance of l formed by the substitution of an individual constant not occurring in e.

[25] The technical definition will be given here. Let l be 'for every x, if x is M, then x is M''. Let l be non-L-false and without variables. Let 'c' be any individual constant not occurring in e; let j say that c is M, and h' that c is M'. Then the qualified-instance confirmation of l with respect to 'M' and 'M'' on the evidence e is defined as follows: $c_{qi}^*('M', 'M'', e) =_{Df} c^*(h', e \cdot j)$.

[26] Some of the theorems may here be given. Let the law l say, as above, that all M are M'. Let 'M_1' be defined, as earlier, by '$M \cdot \sim M'$' ("non-white swan") and 'M_2' by '$M \cdot M'$' ("white swan"). Let the widths of M_1 and M_2 be w_1 and w_2 respectively. Let e be a report about s observed individuals saying that s_1 of them are M_1 and s_2 are M_2, while the remaining ones are $\sim M$ and hence neither M_1 nor M_2. Then the following holds:

$$(1) \qquad c_i^*(l, e) = 1 - \frac{s_1 + w_1}{s + k}.$$

$$(2) \qquad c_{qi}^*('M', 'M'', e) = 1 - \frac{s_1 + w_1}{s_1 + w_1 + s_2 + w_2}.$$

The values of c_i^* and c_{qi}^* for the case that the observed sample does not contain any individuals violating the law l can easily be obtained from the values stated in (1) and (2) by taking $s_1 = 0$.

is a fixed small number, then, with the increase of the sample s, both $\mathfrak{c}_i^*$ and $\mathfrak{c}_{qi}^*$ grow close to 1, in contradistinction to $\mathfrak{c}^*$ for the law itself. This justifies the customary manner of speaking of "very reliable" or "well-founded" or "well confirmed" laws, provided we interpret these phrases as referring to a high value of either of our two concepts just introduced. Understood in this sense, the phrases are not in contradiction to our previous results that the degree of confirmation of a law is very small in a large domain of individuals and 0 in the infinite domain (§13).

These concepts will also be of help in situations of the following kind. Suppose a scientist has observed certain events, which are not sufficiently explained by the known physical laws. Therefore he looks for a new law as an explanation. Suppose he finds two incompatible laws l and l', each of which would explain the observed events satisfactorily. Which of them should he prefer? If the domain of individuals in question is finite, he may take the law with the higher degree of confirmation. In the infinite domain, however, this method of comparison fails, because the degree of confirmation is 0 for either law. Here the concept of instance confirmation (or that of qualified-instance confirmation) will help. If it has a higher value for one of the two laws, then this law will be preferable, if no reasons of another nature are against it.

It is clear that for any deliberate activity predictions are needed, and that these predictions must be "founded upon" or "(inductively) inferred from" past experiences, in some sense of those phrases. Let us examine the situation with the help of the following simplified schema. Suppose a man X wants to make a plan for his actions and, therefore, is interested in the prediction h that c is M'. Suppose further, X has observed (1) that many other things were M and that all of them were also M', let this be formulated in the sentence e; (2) that c is M, let this be j. Thus he knows e and j by observation. The problem is, how does he go from these premisses to the desired conclusion h? It is clear that this cannot be done by deduction; an inductive procedure must be applied. What is this inductive procedure? It is usually explained in the following way. From the evidence e, X infers inductively the law l which says that all M are M'; this inference is supposed to be inductively valid because e contains many positive and no negative instances of the law l; then he infers h ("c is white") from l ("all swans are white") and j ("c is a swan") deductively. Now let us see what the procedure looks like from the point of view of our inductive logic. One might perhaps be tempted to transcribe the usual description of the procedure just given into technical terms as follows. X infers l from e inductively because $\mathfrak{c}^*(l, e)$ is high; since $l \cdot j$ L-implies h, $\mathfrak{c}^*(h, e \cdot j)$ is likewise high; thus h may be inferred inductively from $e \cdot j$. However, this way of reasoning would not be correct, because, under ordinary conditions, $\mathfrak{c}^*(l, e)$ is not high but very low, and even 0 if the domain of individuals is infinite. The difficulty disappears when we realize on the basis of our previous discussions that X does not need a high $\mathfrak{c}^*$ for l in order to obtain the desired high $\mathfrak{c}^*$ for h; all he needs is a high $\mathfrak{c}_{qi}^*$ for l; and this he has by knowing e and j. To put it in another way, X need not take the roundabout way through the law l at all, as is usually believed; he can instead go from his observational knowledge $e \cdot j$ directly to the prediction h. That is to say, our inductive logic makes it possible to determine $\mathfrak{c}^*(h, e \cdot j)$ directly and to find that it has a high value, without making use of any law. Customary thinking in every-day life likewise often takes this short-cut, which is now justified by inductive logic. For instance, suppose somebody asks Mr. *X* what color he expects the next swan he will see to have. Then *X* may reason like this: he has seen many white swans and no non-white swans; therefore he presumes, admittedly not with certainty, that the next swan will likewise be white; and he is willing to bet on it. He does perhaps not even consider the question whether all swans in the universe without a single exception are white; and if he did, he would not be willing to bet on the affirmative answer.

We see that the use of laws is not indispensable for making predictions. Nevertheless it is expedient of course to state universal laws in books on physics, biology, psychology, etc. Although these laws stated by scientists do not have a high degree of confirmation, they have a high qualified-instance confirmation and thus serve us as efficient instruments for finding those highly confirmed singular predictions which we need for guiding our actions.

§15. THE VARIETY OF INSTANCES

A generally accepted and applied rule of scientific method says that for testing a given law we should choose a variety of specimens as great as possible. For instance, in order to test the law that all metals expand by heat, we should examine not only specimens of iron, but of many different metals. It seems clear that a greater variety of instances allows a more effective examination of the law. Suppose three physicists examine the law mentioned; each of them makes one hundred experiments by heating one hundred metal pieces and observing their expansion; the first physicist neglects the rule of variety and takes only pieces of iron; the second follows the rule to a small extent by examining iron and copper pieces; the third satisfies the rule more thoroughly by taking his one hundred specimens from six different metals. Then we should say that the third physicist has confirmed the law by a more thoroughgoing examination than the two other physicists; therefore he has better reasons to declare the law well-founded and to expect that future instances will likewise be found to be in accordance with the law; and in the same way the second physicist has more reasons than the first. Accordingly, if there is at all an adequate concept of degree of confirmation with numerical values, then its value for the law, or for the prediction that a certain number of future instances will fulfill the law, should be higher on the evidence of the report of the third physicist about the positive results of his experiments than for the second physicist, and higher for the second than for the first. Generally speaking, the degree of confirmation of a law on the evidence of a number of confirming experiments should depend not only on the total number of (positive) instances found but also on their variety, i.e. on the way they are distributed among various kinds.

Ernest Nagel[27] has discussed this problem in detail. He explains the difficulties involved in finding a quantitative concept of degree of confirmation that

[27] E. Nagel, *Principles of the Theory of Probability.* Int. Encycl. of Unified Science, vol. I, No. 6, 1939; see pp. 68–71.

would satisfy the requirement we have just discussed, and he therefore expresses his doubt whether such a concept can be found at all. He says (pp. 69f): "It follows, however, that the degree of confirmation for a theory seems to be a function not only of the absolute number of positive instances but also of the kinds of instances and of the relative number in each kind. It is not in general possible, therefore, to order degrees of confirmation in a linear order, because the evidence for theories may not be comparable in accordance with a simple linear schema; and a fortiori degrees of confirmation cannot, in general, be quantized." He illustrates his point by a numerical example. A theory T is examined by a number E of experiments all of which yield positive instances; the specimens tested are taken from two non-overlapping kinds K_1 and K_2. Nine possibilities $P_1, \cdots P_9$ are discussed with different numbers of instances in K_1 and in K_2. The total number E increases from 50 in P_1 to 200 in P_9. In P_1, 50 instances are taken from K_1 and none from K_2; in P_9, 198 from K_1 and 2 from K_2. It does indeed seem difficult to find a concept of degree of confirmation that takes into account in an adequate way not only the absolute number E of instances but also their distribution among the two kinds in the different cases. And I agree with Nagel that this requirement is important. However, I do not think it impossible to satisfy the requirement; in fact, it is satisfied by our concept c*.

This is shown by a theorem in our system of inductive logic, which states the ratio in which the c* of a law l is increased if s new positive instances of one or several different kinds are added by new observations to some former positive instances. The theorem, which is too complicated to be given here, shows that c* is greater under the following conditions: (1) if the total number s of the new instances is greater, *ceteris paribus*; (2) if, with equal numbers s, the number of different kinds from which the instances are taken is greater; (3) if the instances are distributed more evenly among the kinds. Suppose a physicist has made experiments for testing the law l with specimens of various kinds and he wishes to make one more experiment with a new specimen. Then it follows from (2), that the new specimen is best taken from one of those kinds from which so far no specimen has been examined; if there are no such kinds, then we see from (3) that the new specimen should best be taken from one of those kinds which contain the minimum number of instances tested so far. This seems in good agreement with scientific practice. [The above formulations of (2) and (3) hold in the case where all the kinds considered have equal width; in the general and more exact formulation, the increase of c* is shown to be dependent also upon the various widths of the kinds of instances.] The theorem shows further that c* is much more influenced by (2) and (3) than by (1); that is to say, it is much more important to improve the variety of instances than to increase merely their number.

The situation is best illustrated by a numerical example. The computation of the increase of c*, for the nine possible cases discussed by Nagel, under certain plausible assumptions concerning the form of the law l and the widths of the properties involved, leads to the following results. If we arrange the nine possibilities in the order of ascending values of c*, we obtain this: P_1, P_3, P_7, P_9;

P_2, P_4, P_5, P_6, P_8. In this order we find first the four possibilities with a bad distribution among the two kinds, i.e. those where none or only very few (two) of the instances are taken from one of the two kinds, and these four possibilities occur in the order in which they are listed by Nagel; then the five possibilities with a good or fairly good distribution follow, again in the same order as Nagel's. Even for the smallest sample with a good distribution (viz., P_2, with 100 instances, 50 from each of the two kinds) c* is considerably higher—under the assumptions made, more than four times as high—than for the largest sample with a bad distribution (viz. P_9, with 200 instances, divided into 198 and 2). This shows that a good distribution of the instances is much more important than a mere increase in the total number of instances. This is in accordance with Nagel's remark (p. 69): "A large increase in the number of positive instances of one kind may therefore count for less, in the judgment of skilled experimenters, than a small increase in the number of positive instances of another kind."

Thus we see that the concept c* is in satisfactory accordance with the principle of the variety of instances.

§16. THE PROBLEM OF THE JUSTIFICATION OF INDUCTION

Suppose that a theory is offered as a more exact formulation—sometimes called a "rational reconstruction"—of a body of generally accepted but more or less vague beliefs. Then the demand for a justification of this theory may be understood in two different ways. (1) The first, more modest task is to validate the claim that the new theory is a satisfactory reconstruction of the beliefs in question. It must be shown that the statements of the theory are in sufficient agreement with those beliefs; this comparison is possible only on those points where the beliefs are sufficiently precise. The question whether the given beliefs are true or false is here not even raised. (2) The second task is to show the validity of the new theory and thereby of the given beliefs. This is a much deeper going and often much more difficult problem.

For example, Euclid's axiom system of geometry was a rational reconstruction of the beliefs concerning spatial relations which were generally held, based on experience and intuition, and applied in the practices of measuring, surveying, building, etc. Euclid's axiom system was accepted because it was in sufficient agreement with those beliefs and gave a more exact and consistent formulation for them. A critical investigation of the validity, the factual truth, of the axioms and the beliefs was only made more than two thousand years later by Gauss.

Our system of inductive logic, that is, the theory of c* based on the definition of this concept, is intended as a rational reconstruction, restricted to a simple language form, of inductive thinking as customarily applied in everyday life and in science. Since the implicit rules of customary inductive thinking are rather vague, any rational reconstruction contains statements which are neither supported nor rejected by the ways of customary thinking. Therefore, a comparison is possible only on those points where the procedures of customary inductive thinking are precise enough. It seems to me, that on these points sufficient agreement is found to show that our theory is an adequate reconstruction;

this agreement is seen in many theorems, of which a few have been mentioned in this paper.

An entirely different question is the problem of the validity of our or any other proposed system of inductive logic, and thereby of the customary methods of inductive thinking. This is the genuinely philosophical problem of induction. The construction of a systematic inductive logic is an important step towards the solution of the problem, but still only a preliminary step. It is important because without an exact formulation of rules of induction, i.e. theorems on degree of confirmation, it is not clear what exactly is meant by "inductive procedures", and therefore the problem of the validity of these procedures cannot even be raised in precise terms. On the other hand, a construction of inductive logic, although it prepares the way towards a solution of the problem of induction, still does not by itself give a solution.

Older attempts at a justification of induction tried to transform it into a kind of deduction, by adding to the premisses a general assumption of universal form, e.g. the principle of the uniformity of nature. I think there is fairly general agreement today among scientists and philosophers that neither this nor any other way of reducing induction to deduction with the help of a general principle is possible. It is generally acknowledged that induction is fundamentally different from deduction, and that any prediction of a future event reached inductively on the basis of observed events can never have the certainty of a deductive conclusion; and, conversely, the fact that a prediction reached by certain inductive procedures turns out to be false does not show that those inductive procedures were incorrect.

The situation just described has sometimes been characterized by saying that a theoretical justification of induction is not possible, and hence, that there is no problem of induction. However, it would be better to say merely that a justification in the old sense is not possible. Reichenbach[28] was the first to raise the problem of the justification of induction in a new sense and to take the first step towards a positive solution. Although I do not agree with certain other features of Reichenbach's theory of induction, I think it has the merit of having first emphasized these important points with respect to the problem of justification: (1) the decisive justification of an inductive procedure does not consist in its plausibility, i.e., its accordance with customary ways of inductive reasoning, but must refer to its success in some sense; (2) the fact that the truth of the predictions reached by induction cannot be guaranteed does not preclude a justification in a weaker sense; (3) it can be proved (as a purely logical result) that induction leads in the long run to success in a certain sense, provided the world is "predictable" at all, i.e. such that success in that respect is possible. Reichenbach shows that his rule of induction R leads to success in the following sense: R yields in the long run an approximate estimate of the relative frequency in the whole of any given property. Thus suppose that we observe the relative frequencies of a property M in an increasing series of samples, and that we determine on the basis of each sample with the help of the rule R the probability

[28] Hans Reichenbach, *Experience and Prediction*, 1938, §§38 ff., and earlier publications.

q that an unobserved thing has the property M, then the values q thus found approach in the long run the relative frequency of M in the whole. (This is, of course, merely a logical consequence of Reichenbach's definition or rule of induction, not a factual feature of the world.)

I think that the way in which Reichenbach examines and justifies his rule of induction is an important step in the right direction, but only a first step. What remains to be done is to find a procedure for the examination of any given rule of induction in a more thoroughgoing way. To be more specific, Reichenbach is right in the assertion that any procedure which does not possess the characteristic described above (viz. approximation to the relative frequency in the whole) is inferior to his rule of induction. However, his rule, which he calls "the" rule of induction, is far from being the only one possessing that characteristic. The same holds for an infinite number of other rules of induction, e.g., for Laplace's rule of succession (see above, §10; here restricted in a suitable way so as to avoid contradictions), and likewise for the corresponding rule of our theory of $\mathfrak{c}^*$ (as formulated in theorem (1), §10). Thus our inductive logic is justified to the same extent as Reichenbach's rule of induction, as far as the only criterion of justification so far developed goes. (In other respects, our inductive logic covers a much more extensive field than Reichenbach's rule; this can be seen by the theorems on various kinds of inductive inference mentioned in this paper.) However, Reichenbach's rule and the other two rules mentioned yield different numerical values for the probability under discussion, although these values converge for an increasing sample towards the same limit. Therefore we need a more general and stronger method for examining and comparing any two given rules of induction in order to find out which of them has more chance of success. I think we have to measure the success of any given rule of induction by the total balance with respect to a comprehensive system of wagers made according to the given rule. For this task, here formulated in vague terms, there is so far not even an exact formulation; and much further investigation will be needed before a solution can be found.

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