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PROBABILITY AND CONDITIONALS*

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The aim of the paper is to draw a connection between a semantical theory of conditional statements and the theory of conditional probability. First, the probability calculus is interpreted as a semantics for truth functional logic. Absolute probabilities are treated as degrees of rational belief. Conditional probabilities are explicitly defined in terms of absolute probabilities in the familiar way. Second, the probability calculus is extended in order to provide an interpretation for counterfactual probabilities—conditional probabilities where the condition has zero probability. Third, conditional propositions are introduced as propositions whose absolute probability is equal to the conditional probability of the consequent on the antecedent. An axiom system for this conditional connective is recovered from the probabilistic definition. Finally, the primary semantics for this axiom system, presented elsewhere, is related to the probabilistic interpretation.

According to some interpretations of probability theory, a conditional probability statement represents a semantic or pragmatic relation between two propositions. An if–then statement in English, or an analogue in some formal language, also represents a relation between two propositions—the antecedent and the consequent. A lot of philosophical effort has been devoted to the clarification of these two conditional relations, and recently a few philosophers have tried to draw a connection between them.[2] There are at least two reasons motivating the attempts to bring these two problems together. First, although the interpretation of probability is controversial, the abstract calculus is a relatively well defined and well established mathematical theory. In contrast to this, there is little agreement about the logic of conditional sentences. Diverse systems of strict implication, conditional logic, entailment, connexive implication, and causal implication have been proposed and defended on the basis of the vague set of linguistic and methodological intuitions about conditionality, which is all we have to go on. Probability theory could be a source of insight into the formal structure of conditional sentences. Second, one approach to the philosophical problems of induction and confirmation has linked these problems to the analysis of counterfactual conditionals. Other approaches have discussed the problem in the context of interpretations of probability. A connection between the semantics of conditionals and the interpretation of probability might help to bring together the different treatments of these philosophical problems.

In this paper, I shall use probability theory to defend an analysis of conditional propositions which was proposed in another context. My argument has three steps; each step consists of the construction of a probability system. By analogy with

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[2] For some of these discussions, see [4], [1], [2], and [12].

Quine's grades of modal involvement, I might call the systems three grades of conditional involvement, since each is an extension of the preceding one, and with each, conditionality plays a more central role.

In the first system, an absolute probability function is interpreted as an autonomous semantics for propositional calculus, based on the concept of knowledge rather than truth. Conditional probabilities are introduced by definition in the usual way, but are left undefined for some pairs of wffs. The fact that conditional probabilities are sometimes undefined proves a crucial limitation to the system.

The second system provides an interpretation for an extension of the probability calculus in which conditional probabilities are primitive. This system is also an autonomous semantics for propositional calculus based on a concept of *conditional* knowledge.

The third system introduces conditional propositions by adding a primitive conditional connective to the object language and a requirement to the definition of the conditional probability function. The leading idea of the added requirement is that the probability of a conditional statement should equal the conditional probability of the consequent on the antecedent. An axiom system for the conditional connective is then recovered from this probabilistic definition. This system is the formal system of conditional logic, **C2**, which was developed and interpreted independently. I shall conclude the paper by discussing briefly the relation between the probabilistic interpretation of conditional logic and the standard semantics.

1. Absolute Probability Functions. The first system that I shall discuss, $\mathbf{P_1}$, consists of two semantical functions, an absolute probability function and a truth valuation function. I shall first characterize the syntax of the object language, and define these functions. Second, I shall discuss the intuitive content of the functions, and show how one can justify the definition of the probability function in terms of the definition of the truth valuation function. Finally, I shall introduce conditional probabilities as abbreviations, and discuss their interpretation and their limitations.

The primitive symbols of the object language consist of an infinite set of propositional variables, $\{P, Q, R, P', \ldots\}$, two primitive connectives, $\wedge$ and $\sim$ (conjunction and negation, respectively), and parentheses. Any variable is a wff. also, if A and B are wffs, then $\sim A$ and $(A \wedge B)$ are wffs. The additional connectives, $\supset$, $\vee$, $\equiv$ (material conditional, disjunction and material equivalence, respectively) may be defined in terms of the primitives. In this exposition, we shall abbreviate wffs in the usual way.

(1) A *truth valuation function* (tvf) is any function v taking wffs into $\{1, 0\}$ which meets the following two conditions for all wffs A and B:

(a) $v(\sim A) = 1 - v(A)$
(b) $v((A \wedge B)) = v(A) \times v(B)$

(2) An *absolute probability function* (apf) is any function, Pr, taking wffs into real numbers which meets the following six conditions for all wffs A, B, and C:

(a) $1 \geq \Pr(A) \geq 0$
(b) $\Pr(A) = \Pr(A \wedge A)$

(c) $\Pr(A \wedge B) = \Pr(B \wedge A)$
(d) $\Pr(A \wedge (B \wedge C)) = \Pr((A \wedge B) \wedge C)$
(e) $\Pr(A) + \Pr(\sim A) = 1$
(f) $\Pr(A) = \Pr(A \wedge B) + \Pr(A \wedge \sim B)$

(3) A $\mathbf{P}_1$ interpretation is an ordered pair, $\langle v, \Pr\rangle$ where v is a tvf and Pr is an apf, and where for all wffs A, if $\Pr(A) = 1$, then $v(A) = 1$.

A tvf and an apf are two ways to provide an interpretation for wffs, the first in terms of truth and falsity, the second in terms of knowledge and degrees of rational belief. A tvf provides a representation of a *possible world*. Wffs receiving a value of one correspond to propositions which are true in that world, and those with value zero correspond to propositions which are false. An apf provides a representation of a *state of knowledge*.[3] A state of knowledge is here understood to include not only a specification of those propositions known to be true and false, but also a measure of the degree to which the knower has a right to believe propositions which are neither known true nor known false. Values of the function between zero and one exclusive represent the degrees assigned to propositions whose truth value is unknown. Wffs having values of one and zero represent propositions known to be true, and false, respectively.[4]

These two modes of interpretation are not exclusive alternatives, but complementary. A $\mathbf{P}_1$ interpretation combines the two: it provides a representation of a possible world and of the state of knowledge of a knower in that world. The two components of a $\mathbf{P}_1$ interpretation are not completely independent, since a knower cannot know something that is not true. But any probability value between the extremes is compatible with any truth value. Therefore, there is a wide range of apfs which are compatible with any given tvf; this is to say, there may be a diversity of knowers in a single possible world. Also, most apfs are compatible with a variety of tvfs, which is to say that knowledge need not be omniscient: a single state of knowledge may be compatible with many possible states of the world.

For any given state of knowledge, there is a class K of possible worlds which are compatible with that state of knowledge. If the relevant state of knowledge is represented by the apf, Pr, and possible worlds are represented by tvfs, then the class can be defined as follows:

(4) $K =_{\text{df}} \{v / \langle v, \Pr\rangle \text{ is a } \mathbf{P}_1 \text{ interpretation}\}$.

The class K is the class of *epistematically possible worlds*.

[3] More properly, I should say that an apf represents an *idealized* state of knowledge, or a state of *virtual* knowledge, or *implicit* knowledge. I assume that a knower knows implicitly all of the consequences of his knowledge, and more generally, that where A entails B, the degree of rational belief in B is at least as great as that in A. Cf. [3], pp. 31–39.

[4] Some will perhaps be tempted to argue that the identification of knowledge with probability of one is too stringent a condition on knowledge. This temptation should be resisted, since it misses the point of this identification. I am not using a well-established interpretation of probability to provide an analysis of knowledge. Rather, I am using the intuitive notion of knowledge to place constraints on the less clear intuitive notion of probability. No claims about the nature of knowledge are implied by the identification except that knowledge entails truth, and that a state of knowledge is, ideally, deductively closed.

Because an apf is compatible with a range of possible states of the world, it is not possible to define "degree of rational belief" in terms of truth. We can, however, justify all of the constraints on the belief function given in definition (2) in terms of the definition (1) of the tvf. This is accomplished by linking the general concept of degree of rational belief to the general concept of a logically possible world, or a model. This connection is drawn independently of any particular language. In terms of it, the specification of the models for a particular language can be used to evaluate the specific definition of the belief function for that language.

A degree of rational belief in a given proposition for a given subject is interpreted as a number determining the minimum odds which the subject should be willing to accept were he to bet on the truth of that proposition. If $\Pr(A) = r$, then the subject should be willing to bet on A at odds $r/(1 - r)$, and he should be unwilling to accept a bet at odds less favourable than this. The ratio, $r/(1 - r)$ is the ratio of the probability that the proposition is true (that he wins the bet) to the probability that it is false (that he loses the bet). This characterization seems reasonable, since it is reasonable to act on one's beliefs. If you find gambling games a narrow and unsuitable basis on which to build the interpretation of a belief function, consider a "bet" as any action in the face of uncertainty, and the "odds" as the ratio of the value of what you risk by taking the action to the value of what you hope to gain, should the uncertain event turn out in your favor.

A probability assignment to a set of propositions is defined to be *incoherent* if there exists a set of bets for or against those propositions that should be accepted by the subject (according to the assignment), but are such that the subject would sustain a net loss from the set of bets *in every possible outcome*. A probability function is *coherent* if it is not incoherent. If *possible outcomes* are identified with *models of the language*, then we have a general condition of adequacy, stated in terms of the notion of a model, for any belief function. It is obviously reasonable to require that any function determining odds be coherent. If you are willing to accept bets which you are *logically certain* to lose, then you are as irrational as if you had beliefs which are logically certain to be false.

We may use the general definition of coherence to evaluate the system $\mathbf{P}_1$. It can be shown that the conditions defining apf in (2) above are necessary and sufficient to ensure coherence, relative to the class of all models, or tvfs, defined in (1). Every apf is coherent, and every coherent probability function of propositional logic is an apf.[5]

In so far as coherence is our only constraint, the definition of apf is demonstrably correct. But we may still ask, are there further purely logical conditions which should be used to evaluate the adequacy of a definition of belief function? One stronger condition—strict coherence—has been suggested.[6] Strict coherence appears to be a simple and natural strengthening of coherence, and has generally been treated as such. It turns out, however, to require the introduction of some

[5] The notion of coherence was developed by the subjective probability theorists, F. P. Ramsey and Bruno de Finetti. See [6] for the classic papers. For proofs that the probability calculus provides necessary and sufficient conditions for coherence, see [5] and [8].

[6] Strict coherence was first discussed by Abner Shimony in [11].

rather different considerations. Strict coherence is not a *logical* constraint on the belief function, but rather a constraint on the intuitive interpretation of the function, as defined.

A function determining reasonable betting odds is coherent if there is no set of bets consistent with it such that the bettor is certain to suffer a net loss. A function determining betting odds is *strictly* coherent if it is coherent, and also, there is no set of bets consistent with it such that the bettor cannot possibly win, and might lose. The first criterion rules out bets that *must* lose; the second rules out those that *might* lose, and cannot win. This strengthening of coherence seems perfectly reasonable. It is surely irrational to take a risk with no hope of gain, even if there is *some* hope of breaking even.

Kemeny showed, in his paper on fair betting odds, that to ensure that a coherent probability function be strictly coherent, it is necessary and sufficient to add the following requirement:

(5) If $\Pr(A) = 1$, then A is true in all possible outcomes.

The application of this condition depends not only on the truth semantics for the language, but also on an *independent* specification of a class of possible outcomes, or models.[7] Any $\mathbf{P_1}$ interpretation can be shown to be strictly coherent if we take the possible outcomes to be the *epistemically* possible worlds: the situations consistent with the subject's knowledge. This seems reasonable; I take no risk if I bet on the truth of a statement that I know to be true, so I should be willing to accept any odds. And no matter what the odds, I would not bet on something that I know to be false. The set of epistemically possible outcomes is the set K defined in terms of a given apf in (4) above. With K as the set of all possible outcomes, Kemeny's condition (5) follows from the definition of $\mathbf{P_1}$ interpretation. Therefore, we may conclude that every apf is strictly coherent, relative to the set of possible outcomes defined in this way, and that every probability function which is strictly coherent relative to some set of possible outcomes is an apf.

To characterize conditional probabilities in terms of absolute probabilities, we use the familiar definition:

$$(6) \qquad \Pr(A, B) =_{\mathrm{df}} \frac{\Pr(A \wedge B)}{\Pr(B)} \quad (\text{provided } \Pr(B) \neq 0)$$

$\Pr(A, B)$ is undefined when $\Pr(B) = 0$.

Since a conditional probability is simply an abbreviation for a ratio of two absolute probabilities, it is already fully interpreted. We do require, however, a

[7] The requirement, "If A is true in all possible outcomes, then $\Pr(A) = 1$" may be treated as a purely logical constraint, with all possible outcomes interpreted as all tvfs. Then the requirement comes down to "If A is a tautology, $\Pr(A) = 1$, which is entailed by the coherence condition. The converse requirement, however, cannot be treated in the same way without making a host of untenable assumptions. To interpret (5) to mean "If $\Pr(A) = 1$, then A is a tautology," is to confuse a formula with the proposition it represents. Under this interpretation, we should have to accept that every necessary truth—in fact, everything that is known—is a tautology, and that all atomic formulas represent contingent propositions, each of which is logically independent of all the others. If we wish to accept the strict coherence condition without accepting logical atomism, we must allow for an independent specification of a class of models, representing the possible outcomes.

justification for calling this ratio a conditional probability. We can get this justification by giving a separate interpretation to conditional probabilities in terms of odds for conditional bets, and showing that the definition is appropriate to this interpretation.

A conditional bet is a bet that is called off unless a specified condition is met. A bet that P on the condition Q is a bet that is won if P and Q are both true, lost if P is false, and Q is true, and called off is Q is false. A conditional probability is taken as representing reasonable odds for a conditional bet. Where $\Pr(P, Q) = r$, the fair odds for a bet that P on the condition Q are $r/(1 - r)$.

We can justify the definition by showing that such a conditional bet is equivalent to a *pair* of simple bets in the sense that the outcome of the conditional bet (win, lose, or draw) is the same in each possible world as the net outcome of the pair of simple bets. Rather than betting X dollars on the truth of P, conditional on Q, I can achieve the same result by dividing my X dollars in a specifiable way between two bets, one that both P and Q are true, and the other that Q is false. I can always divide the money in such a way that I break even in case I win the second bet (and thus lose the first). For any coherent belief function, if I do divide the money in this way, then I will obtain a net gain or at least break even, should I win the first bet, and lose the second.

Since the two betting situations are equivalent no matter what the outcome, I can determine the fair odds for conditional bets by calculating the ratio of the net gain (in case P and Q are both true) to the net loss (in case P is false and Q true) in the simple betting situation. This calculation gives the same result in every case as the above definition of conditional probability.

Under the intuitive interpretation that we have given to the system $\mathbf{P}_1$, conditionality is given a meaning only when the condition is consistent with the subject's knowledge. In terms of conditional bets, this restriction makes sense: there can be no rational criteria for determining the odds on conditional bets where it is *known* that the condition will remain unfulfilled, and the bet neither won nor lost. This restriction also fits in with some interpretations of conditional *assertions*. Quine, for example, argues that an

> affirmation of the form 'if p then q' is commonly felt less as an affirmation of a conditional than as a conditional affirmation of the consequent. If, after we have made such an affirmation, the antecedent turns out true, then we consider ourselves committed to the consequent, and are ready to acknowledge error if it proves false. If, on the other hand, the antecedent turns out to have been false, our conditional affirmation is as if it had never been made ([10], p. 12).

On this view of conditional assertions, to affirm something on a condition known to be false is to commit oneself to nothing at all, since in such a case it is already known that the affirmation is "as if it had never been made."

Completely excluded by this concept of conditionality, however, is *counterfactual* knowledge, and partial belief. I may believe that *if* Kennedy had not been assassinated, it is highly probable that he would have won the 1964 presidential election. I *know* that the condition is false, but that does not prevent me from speculating—and perhaps speculating rationally—about what would have happened contrary-to-fact. Perhaps I could not place a bet on my counterfactual belief, but this is only because there would be no decisive way of telling who wins. For the same reason, I would not normally bet, say, that no woman will ever run a four-minute mile, or that Moses was actually an Egyptian. For a bet to be practical, there must be an operational decision procedure for determining the truth or falsity of the proposition in question. There must be some expected future event which both I and my gambling opponent would regard as decisively and unambiguously settling the issue. But this is a fact about bets, not about degrees of rational belief or knowledge. The lack of an operational procedure for settling disagreements about what would have been true contrary-to-fact shows not that counterfactual conditional probabilities should not be interpreted, but rather that their interpretation requires an extension of the idea of coherence. Counterfactual assertions are the most controversial and interesting conditional statements. If we are to use probability theory to throw light on these cases, we must first extend the theory to cover counterfactual probabilities. Section **2** presents a generalization of the system $\mathbf{P}_1$ which attempts to do this.

2. Counterfactual Probabilities. The second system $\mathbf{P}_2$, again provides a pair of complementary interpretations to a formulation of classical propositional calculus. The object language, and its primary semantics given by the truth valuation function, are the same as before, but the second semantical function is a conditional probability function. I shall first characterize this function, and $\mathbf{P}_2$ interpretation, and then discuss their intuitive rationale.

(7) An *extended probability function* (*epf*) is any function, Pr, taking ordered pairs of wffs into real numbers which meets the following six conditions for all wffs, A, B, C, and D:

(a) $\Pr(A, B) \geq 0$
(b) $\Pr(A, A) = 1$
(c) If $\Pr(\sim C, C) \neq 1$, then $\Pr(\sim A, C) = 1 - \Pr(A, C)$.
(d) If $\Pr(A, B) = \Pr(B, A) = 1$, then $\Pr(C, A) = \Pr(C, B)$
(e) $\Pr(A \wedge B, C) = \Pr(B \wedge A, C)$
(f) $\Pr(A \wedge B, C) = \Pr(A, C) \times \Pr(B, A \wedge C)$[8]

(8) A $\mathbf{P}_2$ *interpretation* is an ordered pair, $\langle v, \Pr\rangle$, where v is a tvf and Pr is an epf, such that for all wffs A and B, if $\Pr(A, B) = 1$, then $v(B \supset A) = 1$.

[8] The extended probability function is based on one constructed by Sir Karl Popper. Cf. [9], appendix iv. Popper presents his system as an abstract calculus rather than as a semantics. Also, his system has the additional postulate that there must be elements, A, B, C and D such that $\Pr(A, B) \neq \Pr(C, D)$. This has the effect of ruling out the limiting case where $\Pr(A, B) = 1$ for all A and B. In [7], Hughes Leblanc presents two formulations of Popper's system without the added postulate, as a measure on formulas of propositional logic. One of his two formulations is equivalent to the definition of epf.

It should be noted here that Leblanc confuses validity with necessity in the above mentioned article, defining validity so that it is a function of the probability assignment to the variables. Also, the proof that he offers for the equivalence of his two formulations is defective and the equivalence claim is false.

An epf represents an extended state of knowledge. An extended state of knowledge includes, not only a measure of the degree to which the knower has a right to believe certain propositions, but also the degree to which he *would* have a right to believe certain propositions *if* he knew something which in fact he does not know. An epf represents, not just one state of knowledge, but a set of hypothetical states of knowledge, one for each condition. For example, the set of values of $\Pr(A, B)$ for all wffs, A and for a fixed wff B represents the state of knowledge that the knower would be in if he knew B.

Absolute probabilities, are not represented by a primitive function, but they may be defined as a special case of conditional probabilities as follows:

(9) $\Pr(A) =_{\text{df}} \Pr(A, t)$,

where t is some arbitrarily specified tautology.

In the case where the condition is a tautology, conditional knowledge coincides with knowledge *tout court*. It can also be shown that if the condition is known to be true, then the conditional probability is equal to the absolute probability defined in this way. Where $\Pr(B, t) = 1$, $\Pr(A, B) = \Pr(A, t)$ for all A.

Where the condition is itself not known to be true, but also not known to be false, then the conditional state of knowledge will be a function of the actual state of knowledge, exactly as in the classical probability system. An analogue of definition (6), will be a simple consequence of the characterization of epf, (7) together with the above definition of absolute probabilities, (9). In this case, the set of epistemically possible worlds relative to the hypothetical state of knowledge will be a proper subset of the set of epistemically possible worlds, relative to the actual state of knowledge.

When the condition has an absolute probability value of zero, however, the conditional probability values are logically independent of the absolute probability values. Where $\Pr(B) = 0$, $\Pr(A, B)$ may equal zero, one, or anything in between, whatever the absolute probability value of A. In this case, the set of epistemically possible worlds relative to the hypothetical state of knowledge, will be disjoint from the set of epistemically possible worlds relative to the actual state of knowledge.

In the case where the selected state of knowledge is independent of the given one, we require only two things: first, that the resulting hypothetical state of knowledge contain the supposition as an item of knowledge, and second, that the state of knowledge be itself consistent and coherent. For some suppositions, however, it is impossible to meet even these modest requirements. For the supposition may be itself inconsistent or impossible, in which case no coherent state of knowledge can suppose it.

A proposition is an *impossible proposition* if its negation is known true no matter what. A represents an impossible proposition just in case $\Pr(\sim A, A) = 1$. A state of knowledge obtained by assuming an impossible proposition to be true, I shall call an *absurd state of knowledge*. For reasons of determinateness and formal convenience, it is stipulated that where B is impossible, $\Pr(A, B) = 1$ for all A. In the absurd state of knowledge, everything is "known."

An epf, then, is intended to represent an actual state of knowledge and a set of hypothetical states of knowledge, related in a certain way. To show that it succeeds in this intention, I must prove that the constraints set down in the formal definition of an epf are necessary and sufficient for this representation. A few more definitions are needed to make this criterion of adequacy precise.

(10) A bet that A at odds $r/(1-r)$ is *acceptable under condition C* if and only if $\Pr(A, C) \geq r$.

(11) A *conditional* bet that A on condition B at odds $r/(1-r)$ *is acceptable under condition C* if and only if $\Pr(A, B \wedge C) \geq r$.

(12) $K_C^{\Pr} =_{\text{df}} \{v/\text{for all } A, \text{ if } \Pr(A, C) = 1, \text{ then } v(A) = 1\}$.

(13) A knower's probability function, Pr is *strictly coherent with respect to condition C* if and only if there does not exist a set of bets and/or conditional bets acceptable to the knower under condition C such that the knower suffers a net loss in *some* $v \in K_C^{\Pr}$ and a net gain in *no* $v \in K_C^{\Pr}$.

(14) A function Pr is *admissible as an extended belief function* if and only if it is a function taking ordered pairs of wffs into real numbers which meets the following three conditions:

(a) For all wffs C, $v(C) = 1$ for every $v \in K_C^{\Pr}$
(b) Pr is strictly coherent with respect to every C
(c) If $K_C^{\Pr}$ is empty, then $\Pr(A, C) = 1$ for every A.

Definition (10) interprets conditional probabilities not as the odds for a *conditional* bet which are *actually* fair, but rather as the odds for an *unconditional* bet which would hypothetically be fair if the knower were in a different state of knowledge. Definition (11), however, requires that conditional probabilities also represent fair odds for conditional bets—both actual and hypothetical conditional bets. This seems reasonable: the odds that I *would* accept if I knew C to be true for a bet that A should be the same as the odds that I will now accept for a conditional bet that A on condition C.

Definition (12) defines a set of tvfs relative to a belief function Pr and a condition C. This set represents the set of possible worlds that are epistemically possible with respect to the proposition represented by C, or the set of worlds consistent with the hypothetical state of knowledge selected by condition C. Note that where C is a tautology, $K_C^{\Pr}$ represents the set of worlds which are in fact epistemically possible to the knower, and where C is an impossible proposition, $K_C^{\Pr}$ is empty.

Definition (13) is the obvious generalization of the standard definition of strict coherence, and definition (14) states the criterion of adequacy for an extended belief function. Requirement (a) ensures that each hypothetical state of knowledge be the right one—namely one in which the condition C is known to be true. Requirement (b) ensures that each hypothetical state of knowledge meet the same standard of strict coherence that a simple state of knowledge, represented by an apf, must meet. Requirement (c) isolates the absurd state of knowledge and gives the probabilities definite values for it.

Using the results discussed in the first section, I shall sketch a proof of the following theorem:

(15) A function is admissible as an extended belief function if and only it is an epf.

First, the reader can easily verify that for each condition, (a)–(f) of (7), if it is violated, then one of the conditions, (a)–(c) of (14) will be violated. This suffices to prove the first half of the theorem: If a function is admissible as an extended belief function, then it is an epf. To prove the converse, we shall assume that the function, Pr, is an epf and show that each of the three conditions, (a) to (c) of (14) holds.

(a) By (7b), $\Pr(C, C) = 1$ for all C. Therefore, by definition of $K_C^{\Pr}$, for all C, $v(C) = 1$ for every $v \in K_C^{\Pr}$.

(b) Let a function taking single wffs into real numbers be defined for any given C as follows: $\Pr_C(A) =_{\mathrm{df}} \Pr(A, C)$. The function $\Pr_C$ will either be an apf, or else it will be a constant function: $\Pr_C(A) = 1$ for all A. If $\Pr_C$ is an apf, then it will be strictly coherent with respect to the class of tvfs, $\mathrm{K}_C^{\Pr}$. Therefore, in this case, the strict coherence condition is met. If $\Pr_C$ is the constant function, then the strict coherence condition is trivially met, since $K_C^{\Pr}$ is empty.

(c) Finally, if $K_C^{\Pr}$ is empty, then there must be some class of wffs, Γ, such that (i) for all $A \in \Gamma$, $\Pr(A, C) = 1$, and (ii) for every tvf v, there is some $A \in \Gamma$ such that $v(A) = 0$. That is, there is a class of wffs all having probability values of one on the condition C, which is not simultaneously satisfiable. Therefore, by the semantical completeness of propositional calculus, $\Gamma \vdash B$ for all wffs B, from which it follows that for some finite set of wffs, $\{A_1, A_2, \ldots, A_n\}$, all members of Γ, $A_1 \wedge A_2 \wedge \ldots \wedge A_n \vdash B$. But if $\Pr(A_1, C) = \Pr(A_2, C) = \ldots = \Pr(A_n, C) = 1$, then $\Pr(A_1 \wedge A_2 \wedge \ldots \wedge A_n, C) = 1$. Therefore, since the probability of a proposition is always equal to or greater than the probability of something that entails it, $\Pr(B, C) = 1$ for all wffs B. This completes the proof.

To conclude this section I wish to contrast the intuitive content of the extended probability system with that of the standard system. What is the nature of the additional information which would be contained in an extended system? A classical probability function as I have interpreted it, provides a measure of the simple epistemological status of propositions. Things are better or less well known according as their probability values are greater or less. The standard function does not, however, make any distinctions among propositions which are known to be true, and it can say nothing about the relations between propositions which are known to be true. Mathematical theorems may be ranked with empirical hypotheses. Simple facts are not distinguished from basic scientific principles. And one statement may be evidence for another, or independent of it, without this difference being reflected in the probability values. An extended function, on the other hand, contains information which is relevant to these differences in at least three ways:

First, an epf distinguishes between items of knowledge which are contingent and items of knowledge which are necessary. The former are *merely* known, while the latter *would* be known in all states of knowledge, or under every supposition. That is, A is a necessary truth if $\Pr(A, C) = 1$ for all C. What would be known under any condition is the same as what is true in all possible worlds, where the set, K of possible worlds is defined as the union of the sets K_C^{Pr} for all C. A world is ontologically possible if it is epistemologically possible relative to some supposition.

Second, an epf allows for a distinction between superficial facts—things we just happen to know—and items of empirical knowledge which have profound systematic interconnections with other parts of our knowledge. A superficial bit of information is an item of knowledge which would easily be called into question by counterfactual suppositions, and which could be hypothetically denied with only minor changes in the state of knowledge. An entrenched systematically important truth, on the other hand, would remain an item of knowledge under diverse counterfactual suppositions, and its hypothetical denial would force a radical change in the state of knowledge.

Third, an epf contains some information about the inductive relations among propositions known to be true. If there is a strong correlation between the rise and fall of the probability values of A and B under different counterfactual assumptions, for example, then one could conclude that the events described by A and B were causally connected in some way. By looking at the values of $\Pr(A, C)$ and $\Pr(B, C)$ for various particular C's, one might determine *how* they were causally connected.

In general, counterfactual suppositions allow us to go beneath the surface of our knowledge in order to get at both the inductive and the conceptual relations among the things that we know, or believe to various degrees. The rules defining an epf do not, of course, provide any procedures for answering questions about these underlying relations, any more than logic provides criteria for truth. They do, however, offer a framework in which the counterfactual beliefs, which we undoubtedly have and use, can be represented.

In the final section, I shall extend this system by introducing conditional *propositions*. This will make it possible for the inductive and conceptual relations reflected in an extended probability system to be represented as explicit beliefs and items of knowledge.

3. Conditional Propositions. The third system, $\mathbf{P}_3$, involves not only an extension of the probability function defined in section **2**, but also a change in the object language, and the truth semantics. In defining this system, I shall proceed somewhat differently than in the first two cases. First, I shall describe the syntax of the new object language, **C2**. Second, I shall add a requirement to the definition of probability function which establishes a connection between conditional propositions and conditional probabilities. Third, I shall ask what logical properties conditional propositions must have in order that the probability function have the form that it does have. Thus, our procedure is here the reverse of what it was in sections **1** and **2**. In those sections, the established primary semantics was used, in conjunction with an idea of coherence, to justify the probability semantics. In this section, a natural extension of the probability semantics in conjunction with the idea of coherence will be used to discover and justify the rules of truth for conditional propositions.

The object language, **C2**, is as before except that one connective, $>$ (called the corner) is added to the list of primitive symbols, and one clause is added to the definition of wff as follows: if A and B are wffs, then $(A > B)$ is a wff.

A **C2**-epf is defined as a function taking ordered pairs of wffs of **C2** into real numbers. The function must meet all of the requirements of an ordinary epf, as set down in definition (7), section **2** above. It must also meet one additional requirement. Our first problem is to determine exactly what that should be.

The absolute probability of a conditional proposition—a proposition of the form $A > B$—must be equal to the conditional probability of the consequent on the condition of the antecedent.

(16) $\quad \Pr(A > B) = \Pr(B, A)$

The probability of the proposition, if Nixon is nominated then Johnson will win, should be the same as the probability that Johnson will win, on the condition that Nixon is nominated. This is the basic requirement, but by itself it is too weak, since it sets no limits on the *conditional* probability of conditional propositions. On the basis of the requirement (16), we could draw certain conclusions about the *absolute* probabilities of conditional propositions—for example that for all wffs A and B, $\Pr(A > B) = 1 - \Pr(A > \sim B)$ whenever $\Pr(A > \sim A) = 0$. But we could draw no conclusion at all about conditional probabilities of conditionals. For example, for any wffs C such that $\Pr(C) < 1$, the relation between $\Pr(A > B, C)$ and $\Pr(A > \sim B, C)$ would be completely open. Thus no real constraints would be placed on the logic of conditionals since any set of conditional formulas would be simultaneously satisfiable in the sense that there would exist a **C2**-epf which assigned each formula the value one on some consistent condition.

The following generalization of the proposed requirement suggests itself:

(17) $\quad \Pr(A > B, C) = \Pr(B, A \wedge C)$

This condition, however, is clearly too strong.

The antecedent A may be a *counterfactual* assumption with respect to the condition C. That is, the antecedent A may be incompatible with the state of knowledge selected by the condition C. In this case the antecedent A cannot simply be added to the set of things known in that state of knowledge. Some deletions and adjustments will have to be made, and the condition C may be one of the things that gets deleted. In fact, the adoption of the strong requirement, (17) would trivially give all counterfactual propositions a probability of one, collapsing the distinction between knowledge and necessity, and reducing the probability system, $\mathbf{P}_3$ to one roughly equivalent to $\mathbf{P}_1$. This can be seen by the following argument: suppose $A > B$ represents a counterfactual—that is a conditional proposition whose antecedent is known to be false. Then $\Pr(A) = 0$, so $\Pr(\sim A) = 1$. But for all C such that $\Pr(C) = 1$, and for all D, $\Pr(D, C) = \Pr(D)$. Therefore $\Pr(A > B) = \Pr(A > B, \sim A)$. But by requirement (17), $\Pr(A > B, \sim A) = \Pr(B, A \wedge \sim A)$, which always equals one. Therefore $\Pr(A > B) = 1$. But all we assumed was that $A > B$ was counterfactual.

In order to steer a course between the unacceptably weak condition (16) and the unacceptably strong (17), we must generalize (16) in a different way. In order to carry out this generalization, we need a few more definitions.

(18) A function, Pr_C, taking ordered pairs of wffs of **C2** into real numbers is a *subfunction of Pr with respect to C* iff Pr is also a function taking pairs of wffs into real numbers, C is a wff of **C2**, and for all wffs A and B, $\mathrm{Pr}_C(B, A) = \mathrm{Pr}(A > B, C)$.

(19) A function Pr taking ordered pairs of wffs of **C2** into real numbers is *acceptable on the first level* iff it is an epf and for all wffs A and B, $\mathrm{Pr}(A > B) = \mathrm{Pr}(B, A)$.

(20) A function Pr taking pairs of wffs of **C2** into real numbers is *acceptable on the $(n + 1)$th level* if for every wff C, the subfunction of Pr with respect to C is acceptable on the nth level.

(21) A function Pr taking pairs of wffs of **C2** into real numbers is a **C2**-*epf* if it is acceptable on the nth level for every n.

The introduction of subfunctions is simply a device to allow the weak requirement (16) to be applied more generally without collapsing conditions as does the rejected requirement (17).

Definition (21) gives a complete semantical characterization of a conditional concept, not in terms of its truth relations, but in terms of its probability relations. The next step in the investigation is to define notions of satisfiability and validity for the wffs of the language **C2**, relative to this probability semantics. Then I shall present an axiom system which implicitly defines syntactical notions of consistency and theoremhood for conditional logic. This system, will then be proved semantically sound and complete relative to the probability semantics.

The final step of the argument—the construction of an appropriate truth semantics—has already been taken. The axiom system for **C2** has elsewhere been shown to be semantically sound and complete relative to a primary semantics which was given an independent philosophical justification.[9] In the conclusion to this paper, I shall discuss the relation between the two semantical systems.

(22) A class Γ of wffs of **C2** is *p-simultaneously satisfiable* if there exists a **C2**-epf Pr and a wff C such that $\mathrm{Pr}(\sim C, C) \neq 1$, and for all $A \in \Gamma$, $\mathrm{Pr}(A, C) = 1$.

(23) A wff A is *p-valid* if $(\sim A)$ is not p-simultaneously satisfiable.

A simultaneously satisfiable class, by this definition, represents a class of propositions, all of whose members might be known to be true. A valid formula represents a proposition whose negation could not possibly be known to be true.

To specify the formal system, I shall use two nonprimitive modal operators, defined as follows:

(24) Definition schemata:
(a) $\Box A =_{\mathrm{df}} \sim A > A$
(b) $\Diamond A =_{\mathrm{df}} \sim \Box \sim A$

[9] The completeness proof is presented in [14]; [13] is an informal exposition and philosophical defense of the theory.

These definitions bring out the fact that by moving from conditional probabilities to conditional propositions, we have also implicitly moved from a modal predicate of propositions, in the meta-language, to a modal operator, in the object language. In Quine's terminology, we have moved from the first to the second grade of modal involvement. In the system, $\mathbf{P}_2$, $\Pr(A, \sim A) = 1$ just in case A is a necessary truth. Therefore, in $\mathbf{P}_3$, we have a *proposition* which states that A is a necessary truth.

The following two rules and seven axiom schemata determine the formal system **C2**:

(25) Rules:

(a) If $A \supset B$ and A are theorems, then B is a theorem
(b) If A is a theorem then $\Box A$ is a theorem

(26) Axiom schemata:

(a) Any tautologous wff is an axiom
(b) $\Box(A \supset B) \supset \cdot \Box A \supset \Box B$
(c) $\Box(A \supset B) \supset \cdot A > B$
(d) $\Diamond A \supset \cdot A > B \supset \sim(A > \sim B)$
(e) $A > (B \vee C) \supset . (A > B) \vee (A > C)$
(f) $A > B \supset . A \supset B$
(g) $(A > B) \wedge (B > A) \supset . (A > C) \supset . (B > C)$

In the usual way, these rules and axioms determine the syntactical notions, **C2**-provability, **C2**-derivability and **C2**-consistency.

Before stating the semantical completeness theorem, I shall list some object language theorem schemata which will be useful in the metaproof.

(27) Theorem schemata:

(a) $\vdash (\mathrm{t} > A) \equiv A$ (where t is any tautology)
(b) $\vdash A > A$
(c) $\vdash \Diamond C \supset . (C > A) \equiv \sim(C > \sim A)$
(d) $\vdash C > (A \wedge B) \equiv (C > (B \wedge A))$
(e) $\vdash C > (A \wedge B) \equiv ((C > A) \wedge ((A \wedge C) > B))$

We are now equipped to sketch a proof of the following semantical completeness theorem:

(28) A class Γ of wffs of **C2** is p-simultaneously satisfiable if and only if it is **C2**-consistent.

The first half of the proof consists of validating the axioms and showing that the rules preserve validity. First, note that for any wffs A and C, if there exists a **C2**-epf Pr which satisfies A on condition C then there exists a **C2**-epf which satisfies A on condition t (where t is a tautology), namely the subfunction of Pr, $\Pr_C$. Therefore, to validate an axiom, it suffices to show that it is not satisfiable on condition t. Second, note that every axiom, in unabbreviated form, is the negation of a conjunction. For each axiom, assume that this conjunction has an absolute probability value of one (that is, assume that the negation of the axiom is satisfiable on condition t). In each case, a contradiction will fall out relatively easily. To show that modus ponens, (25a) preserves validity, assume that $\Pr(A \supset B) = 1$ and $\Pr(A) = 1$ for all **C2**-epfs. Then $\Pr(A \wedge \sim B) = \Pr(A) \times \Pr(\sim B, A) = \Pr(\sim B, A) = 0$. So $\Pr(B, A) = 1$. But since $\Pr(A) = 1$, $\Pr(B, A) = \Pr(B)$, so $\Pr(B) = 1$ in all **C2**-epfs. To show that the necessitation rule, (25b) preserves validity, assume A is valid. Then $\{\sim A\}$ is not p-satisfiable, so for all **C2**-epfs Pr and wffs C such that $\Pr(\sim C, C) \neq 1$, $\Pr(\sim A, C) < 1$. But for all **C2**-epf's, Pr, $\Pr(\sim A, \sim A) = 1$, so to avoid contradiction we must conclude that $\Pr(\sim\sim A, \sim A) = 1$, and hence that $\Pr(A, \sim A) = 1$ for all **C2**-epfs Pr. Therefore $\Pr(\sim A > A)$, which is the same as $\Pr(\Box A)$, must be equal to one. So both rules preserve validity.

To prove the converse, I shall show that given any **C2**-consistent class of wffs, Γ, it is possible to construct a **C2**-epf and a wff C such that $\Pr(\sim C, C) \neq 1$ and $\Pr(A, C) = 1$ for all $A \in \Gamma$. The argument follows the familiar method developed by Henkin. First, in the usual manner, construct a maximally consistent class, Γ^* which contains Γ. Then let a function, Pr, taking ordered pairs of wffs into real numbers be defined as follows: For all wffs A and B, $\Pr(A, B) = 1$ if $(B > A) \in \Gamma^*$, and $\Pr(A, B) = 0$ otherwise. Let C be an arbitrarily selected tautology, t. Substituting $\sim$t for A in theorem (27a), we get $\vdash(\mathrm{t} > \sim\mathrm{t}) \equiv \sim\mathrm{t}$. Since Γ^* is consistent, $\sim\mathrm{t} \notin \Gamma^*$, and therefore $\mathrm{t} > \sim\mathrm{t} \notin \Gamma^*$, so $\Pr(\sim\mathrm{t}, \mathrm{t}) = 0$. Also by theorem, (27a) and the consistency of Γ^*, it is evident that $\Pr(A, \mathrm{t}) = 1$ iff $A \in \Gamma^*$. Since $\Gamma \subseteq \Gamma^*$, $\Pr(A, \mathrm{t}) = 1$ for all $A \in \Gamma$. Therefore, the function, Pr and the wff C that we have constructed meet the conditions of definition (22). It remains only to show that the function Pr is a **C2**-epf. This we shall do by going through the six defining requirements for epf given in (7), and the added requirement for **C2**-epf given in (21).

(a) $\Pr(A, B) = 0$ or 1 for all A and B, so $\Pr(A, B) \geq 0$.

(b) $\vdash A > A$ by (27b), so $\Pr(A, A) = 1$ for all A.

(c) Assume $\Pr(\sim C, C) \neq 1$. Then $\Pr(\sim(C > \sim C), \mathrm{t}) = \Pr(\Diamond C, \mathrm{t}) = 1$, so $\Diamond C \in \Gamma^*$. Then by (27c), $(C > A) \equiv \sim(C > \sim A) \in \Gamma^*$. Therefore $\Pr(A, C) = 1(0)$ iff $\Pr(\sim A, C) = 0(1)$. Hence provided $\Pr(\sim C, C) \neq 1$, $\Pr(\sim A, C) = 1 - \Pr(A, C)$.

(d) Assume $\Pr(A, B) = \Pr(B, A) = 1$. In this case, $A > B \in \Gamma^*$ and $B > A \in \Gamma^*$. Therefore, by an axiom, (26g), $(A > C) \equiv (B > C) \in \Gamma^*$, so $\Pr(C, A) = \Pr(C, B)$, provided $\Pr(A, B) = \Pr(B, A) = 1$.

(e) By (27d), $C > (A \wedge B) \in \Gamma^*$ iff $C > (B \wedge A) \in \Gamma^*$, so $\Pr(A \wedge B, C) = \Pr(B \wedge A, C)$.

(f) By (27e), $C > (A \wedge B) \in \Gamma^*$ iff $C > A \in \Gamma^*$ and $(A \wedge C) > B \in \Gamma^*$. Therefore, $\Pr(A \wedge B, C) = 1$ iff $\Pr(A, C) = 1$ and $\Pr(B, A \wedge C) = 1$. Therefore, $\Pr(A \wedge B, C) = \Pr(A, C) \times \Pr(B, A \wedge C)$.

(g) That the function Pr is acceptable on the first level follows from a special case of (27a) $\vdash \mathrm{t} > (A > B) \equiv (A > B)$.

(h) To show the function acceptable on the n-th level, in general, it suffices to show that every subfunction of Pr, and subfunction of a subfunction of Pr, etc. meets the first six conditions, and that each is acceptable on the first level. We do this by generalizing each of the above seven arguments. Using the following derived rules and distribution principles, the generalizations are quite straightforward, although in a few cases tedious.

(29) Derived rules and theorems schemata

(a) If $\vdash A$, then $\vdash C_1 > (C_2 > \ldots > (C_n > A)$.

(b) If $\vdash A \supset B$, then $\vdash (C > A) \supset (C > B)$

(c) $\vdash C > (A \equiv B) \equiv ((C > A) \equiv (C > B))$

(d) $\vdash C > (A \wedge B) \equiv (C > A) \wedge (C > B)$

These generalizations complete the argument. The function Pr is a **C2**-epf, and thus the arbitrary consistent class Γ is p-satisfiable.

4. Possible Worlds and Knowledge. In conclusion, I shall explain briefly the intuitive idea behind the primary sematics for **C2** and consider the relation between this system and the one based on probability that I have been discussing.

A conditional statement, according to a theory of conditionals that I have defended elsewhere, is a statement about a particular possible world. *Which* possible world it is about is a function of the antecedent. Which statement is made about that world is a function of the consequent. The particular possible world selected by the antecedent cannot be just any world. First, it must be one in which the antecedent is true; when we say "if A . . ." We are supposing A to be *true.* Second, it must resemble the actual world as closely as possible, given the first requirement. This latter restriction means that, where the antecedent is true in the actual world, the actual world is the world I am talking about. That is why when one asserts a conditional which turns out to have a true antecedent, he is committed to the consequent. The latter restriction also means that the world selected carries over as much of the explanatory and descriptive structure of the actual world as is consistent with the antecedent. That is why causal laws and well entrenched empirical relations are relevant to the evaluation of a counterfactual.

These intuitive ideas can be represented in a semantical theory for a formal language which includes a primitive conditional connective. An interpretation of a set of formulas is defined on a *model structure* which consists of a structure of possible worlds. The interpretation on the structure is relative to a *selection function*, *f*—a function that selects, for each formula A and possible world α a possible world in which A is true. The truth rule for conditional formulas—formulas of the form $(A > B)$—can be stated as follows:

(30) For all wffs A and B, and all possible worlds α, $(A > B)$ is true in α iff B is true in $f(A, \alpha)$.

These truth conditions, together with constraints on the selection function which are appropriate to the intuitive picture sketched above, give rise to semantical concepts of satisfiability and validity for the formulas of **C2**. The axiom system given in section **3** is sound and complete with respect to these concepts.

According to this semantical theory, the evaluation of a conditional statement involves, implicitly, the weighing of possible worlds against each other. To decide about a conditional, I must answer a hypothetical question about how I would revise my beliefs in the face of a particular potential discovery. We are all, of course, continually making such revisions, both actual and hypothetical, and this process of change reflects methodological patterns and principles. There are always alternative ways to patch up our structure of beliefs, as Quine has persuasively argued, but the choice among the alternatives is not arbitrary. Some opinions acquire a healthy immunity to contrary evidence and become the core of our conceptual system, while others remain near the surface, vulnerable to slight shifts in the phenomena. The policies by which we make distinctions like this lend some stability to the changing process of inquiry.

A selection function, selecting and ordering possible worlds, is intended as a representation of these methodological policies. A probability system is also a representation of them, since the same policies would be involved in the determination of degrees of belief. The difference is that a probability system represents in addition the limited perspective of an individual knower. The move through the various grades of conditional involvement—$\mathbf{P}_1$ to $\mathbf{P}_3$—is an attempt to sort out the general principles from the factors that depend on the particular part of the actual world a knower has experienced, or learned about. The primary semantics for **C2** is the final step in this sorting out.

My intention in developing these formal and intuitive parallels between the theory of conditional probability and the semantics for conditional logic has been to give some additional support to the analysis of conditional statements sketched above. Beyond this, it is hoped that with the further development of the theories (for example the addition of quantifiers), this approach may provide some tools for the philosophical analysis of induction and confirmation.

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