2. Reformulations

2.1 Multiple Modalities

There are various ways to formulate my analysis of counterfactuals as variably strict conditionals based on comparative similarity of worlds. Let us look at some of the alternative formulations. Some are exactly equivalent to my first formulation by means of systems of spheres; others are equivalent only to special cases thereof.

Suppose there are no more than a certain finite number n of nonempty spheres around any world. Then we can number the spheres around each world i in order of increasing size. We begin with S_i^0, the empty set; then comes S_i^1, the innermost nonempty sphere (assuming centering, S_i^1 is $\{i\}$); then S_i^2, the next smallest; and so on out to S_i^n, the largest sphere around i. (In case i has fewer than its full complement of n distinct nonempty spheres, we give all the left-over numbers to the outermost sphere. If there are only $n-2$ nonempty spheres around a certain world i, for instance, the outermost of them counts as S_i^{n-2}, as S_i^{n-1}, and as S_i^n.) We introduce a family of increasingly strict necessity operators $\Box_1, \ldots, \Box_n$, together with the corresponding possibility operators $\Diamond_1, \ldots, \Diamond_n$. For any number m from 1 through n, $\Box_m\phi$ is to be true at a world i if and only if ϕ holds throughout S_i^m; and $\Diamond_m\phi$ is to be true at i and if only if ϕ holds at some world in S_i^m. In other words, the spheres S_i^m are the spheres of accessibility for the mth pair of modal operators, $\Box_m$ and $\Diamond_m$.*

Given such a family of modalities, the counterfactual connectives are definable.

$$\phi \;\Box\!\!\rightarrow\; \psi =^{df} (\Diamond_1\phi \;\&\; \Box_1(\phi \supset \psi)) \vee \ldots \vee (\Diamond_n\phi \;\&\; \Box_n(\phi \supset \psi)) \vee \;\sim\Diamond_n\phi,$$

$$\phi \;\Diamond\!\!\rightarrow\; \psi =^{df} (\Diamond_1\phi \supset \Diamond_1(\phi \;\&\; \psi)) \;\&\ldots\&\; (\Diamond_n\phi \supset \Diamond_n(\phi \;\&\; \psi)) \;\&\; \Diamond_n\phi.$$

* Such a system of multiple modalities is discussed in M. K. Rennie, 'Models for Multiply Modal Systems', *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* **16** (1970): 175–186, and in L. F. Goble, 'Grades of Modality', *Logique et Analyse* **51** (1970): 323–334.

More generally, if we were willing to assume that there are no infinite descending sequences of smaller and smaller spheres around any one world, we could number the spheres around each world by ordinals, including perhaps transfinite ordinals. We could then introduce an infinite family of increasingly strict necessity operators, together with the corresponding possibility operators, indexed by ordinals and thereby placed in correspondence with the spheres around each world. We could then define the counterfactual connective by an infinite disjunction (for $\Box\!\!\rightarrow$) or an infinite conjunction (for $\Diamond\!\!\rightarrow$) of disjuncts or conjuncts like those in the finite definitions above.

2.2 Propositional Quantification

Infinite disjunctions or conjunctions often can be replaced by existential or universal quantifications. They can be thus replaced in this case; and as a bonus we can drop the restriction against infinite descending sequences of smaller and smaller spheres around a world. We could quantify over modalities themselves;* instead of a disjunction or conjunction of parallel clauses involving different modalities, we could have a definiens in which an initial quantifier over suitable modalities binds a modal-operator-variable in its scope. But for our present purposes we need nothing so exotic. Propositional quantification will serve as well.

Suppose that our language has the following resources: (1) propositional variables, grammatically interchangeable with sentences; (2) existential and universal quantifiers $\exists$ and $\forall$ that may be used to bind these variables; (3) the operators $\Box$ and $\Diamond$ of outer necessity and possibility; (4) the truth-functional connectives; and (5) a special one-place sentential operator $\bigcirc$, called the *sphericality* operator. A sentence $\bigcirc\phi$ is to be true at a world i if and only if there is some sphere S around i such that ϕ is true at all and only the worlds in S. More precisely, since ϕ may be an open sentence with free propositional variables: $\bigcirc\phi$ is true at i, relative to a given assignment of values to its free propositional variables (if any), if and only if there is some sphere S around i such that ϕ is true, relative to that assignment of values, at all and only the worlds in S.‡

Now we are ready to define the counterfactual connectives. We have

* A system that permits quantification over modalities is given in Richard Montague, 'Universal Grammar', *Theoria* 36 (1970): 373–398; reprinted in Montague, *Formal Philosophy*.

‡ Alternatively, suppose we are given the logical modalities rather than the outer modalities; then we may begin by defining the outer modalities using the given apparatus.

only to copy their truth conditions into the object language. We no longer assume any special restrictions on the system of spheres.

$$\phi \;\Box\!\!\rightarrow \psi =^{\mathrm{df}} \Diamond\phi \supset \exists\xi(\bigcirc\xi \;\&\; \Diamond(\xi \;\&\; \phi) \;\&\; \Box(\xi \;\&\; \phi. \supset \psi)),$$
$$\phi \Diamond\!\!\rightarrow \psi =^{\mathrm{df}} \Diamond\phi \;\&\; \forall\xi(\bigcirc\xi \;\&\; \Diamond(\xi \;\&\; \phi). \supset \Diamond(\xi \;\&\; \phi \;\&\; \psi)).$$

Here ξ is any variable that does not occur in ϕ or ψ.

The values of propositional variables are, of course, called *propositions*. It does not much matter what propositions are, so long as (1) they are entities that can be true or false at worlds, and (2) there are enough of them. They must have truth values at worlds so that an open sentence consisting of a propositional variable standing alone will have truth values at worlds, relative to an assignment of a value to the variable: the truth value of the sentence is the truth value of the proposition assigned as value to the variable. For every proposition, as for every sentence, there is a set of the worlds where it is true. Conversely, for each set of worlds, there should be a proposition true at all and only the worlds in that set. Otherwise we cannot safely transform quantification over sets of worlds in the metalanguage into propositional quantification in the object language, as we did to obtain our definitions of the counterfactual operators.

For the sake of definiteness, we may take sets of worlds to *be* propositions.* A proposition P is true at a world i if and only if i belongs to the proposition—the set—P. There is a proposition for every set of worlds because the set itself is the proposition true at all and only the worlds

* As is done in much recent work in possible-world semantics. (Sometimes with a trivial difference: propositions are taken to be the characteristic functions of sets of worlds rather than the sets themselves.) The idea goes back at least to Clarence I. Lewis, 'The Modes of Meaning', *Philosophy and Phenomenological Research* 4 (1944): 236–249, in which the set of worlds is called the 'comprehension' of the proposition; and to Rudolf Carnap, *Meaning and Necessity* (University of Chicago Press: Chicago, 1947), in which propositions are taken as sets of state descriptions, and state descriptions are said to 'represent Leibniz' possible worlds or Wittgenstein's possible states of affairs'. No theory can fit all that philosophers have said about 'propositions'—they have said too much—but the identification of propositions with sets of worlds captures a good part of the tradition. Propositions so understood are non-linguistic entities capable of being true or false. They exist eternally, non-contingently, and independently of us. One proposition may be expressed by many sentences, in one language or in many, or by non-verbal means of communication; on the other hand, there may be propositions that we have no way to express. Two sentences that are logically equivalent, or that do not differ in truth value at any world for whatever reason, express the same proposition. But one part of the tradition about propositions must be given up: propositions understood as sets of worlds cannot serve as the meanings of sentences that express them, since there are sentences—for instance, all the logical truths—that express the same proposition but do not, in any ordinary sense, have the same meaning.

in the set. For any sentence ϕ, let $[\![\phi]\!]$ be the set of worlds where ϕ is true. $[\![\phi]\!]$, being a set of worlds, is a proposition; call it the proposition *expressed* by the sentence ϕ. Then a sentence ϕ is true at a world i if and only if the proposition $[\![\phi]\!]$ expressed by ϕ is true at i; that is, if and only if i belongs to the proposition $[\![\phi]\!]$. All the tautologies express the same proposition: the *necessary proposition,* in other words the set of all worlds. All contradictions express the same proposition: the *impossible proposition,* in other words the empty set. A proposition expressed by some or other sentence of a language is said to be *expressible* in that language. We cannot safely assume that every proposition is expressible in our language, or indeed in any practical enrichment thereof. There are apt to be too many propositions and too few sentences. (I shall argue in Section 4.1 that there are more worlds than sets of sentences. *A fortiori* there are more propositions than sentences.) That is why we need to quantify over propositions. Quantification over sentences—in effect, over expressible propositions—could not substitute for meta-linguistic quantification over sets of worlds.

If sets of worlds are propositions, the truth conditions for many sentential connectives and operators can be restated by means of an algebra of propositions. With an n-place connective we associate an n-place operation on propositions, so that the proposition expressed by a compound sentence is obtained by applying the operation to the propositions expressed by the sentences whence it was compounded. Negation corresponds to complementation relative to the set I of all worlds; conjunction to intersection; disjunction to union; and so on for the other truth functions. Then the truth conditions for compound sentences are given by propositional equations:

$$[\![\sim\phi]\!] = I - [\![\phi]\!],$$
$$[\![\phi \;\&\; \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!],$$
$$[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!].$$

Our counterfactual connective $\Box\!\!\rightarrow$ corresponds to a more complicated two-place operation on propositions; call it the *counterfactual operation.* Given as arguments two sets of worlds P and Q, this operation yields as value the set of all worlds i such that if P overlaps any sphere around i, then P overlaps some sphere S around i such that the intersection $P \cap S$ is included in Q. We can now state the truth conditions for counterfactuals by saying that, for any ϕ and ψ, the proposition $[\![\phi \Box\!\!\rightarrow \psi]\!]$ is the result of applying this counterfactual operation to the propositions $[\![\phi]\!]$ and $[\![\psi]\!]$. We can say that the connective *expresses* the operation. If we want to give the connective an entity to be its meaning, the operation can serve the purpose.

2.3 *Comparative Similarity*

Our system of spheres is nothing but a convenient device for carrying information about the comparative similarity of worlds. We could do away with the spheres, and give the truth conditions for counterfactuals directly in terms of comparative similarity of worlds, together with accessibility. Let us introduce the notation

$$j \leqslant_i k$$

to mean that the world j is at least as similar to the world i as the world k is; also

$$j <_i k \quad \text{(defined as: it is not the case that } k \leqslant_i j\text{)}$$

to mean that j is more similar to i than k is. We may posit an assignment to each world i of two items: a two-place relation $\leqslant_i$ among worlds, regarded as the ordering of worlds in respect of their comparative similarity to i, and a set S_i of worlds, regarded as the set of worlds accessible from i. Call such an assignment a *(centered) comparative similarity system* if and only if, for each world i, the following six conditions hold.

(1) The relation $\leqslant_i$ is *transitive*; that is, whenever $j \leqslant_i k$ and $k \leqslant_i h$, then $j \leqslant_i h$.
(2) The relation $\leqslant_i$ is *strongly connected*; that is, for any worlds j and k, either $j \leqslant_i k$ or $k \leqslant_i j$. (Equivalently: if $j <_i k$ then $j \leqslant_i k$.)
(3) The world i is *self-accessible*; that is, i belongs to S_i.
(4) The world i is *strictly $\leqslant_i$-minimal*; that is, for any world j different from i, $i <_i j$.
(5) Inaccessible worlds are *$\leqslant_i$-maximal*; that is, if k does not belong to S_i, then for any world j, $j \leqslant_i k$.
(6) Accessible worlds are more similar to i than inaccessible worlds: if j belongs to S_i and k does not, then $j <_i k$.

A relation that is transitive and strongly connected is called a *weak ordering* or a *(total) preordering.** We can state the six conditions concisely as follows: each $\leqslant_i$ is a weak ordering of the worlds, with i alone at the bottom and all the worlds inaccessible from i, if there are any, together at the top above all the accessible worlds. All inaccessible

* 'Weak' because, unlike a *strong* (or *linear*) *ordering*, ties are permitted: two different things can stand in the relation to each other, and thus be tied in the ordering. 'Preordering' because if we take equivalence classes under the relation of being thus tied, the induced ordering of the equivalence classes is a strong ordering. Familiar weak orderings are the relations of being at least as tall as, at least as far north as, etc. When I speak simply of an *ordering*, I shall mean a weak ordering; we shall be little concerned with strong orderings.

worlds are equally dissimilar to i; if j and k both are outside S_i, then $j \leqslant_i k$ and $k \leqslant_i j$. If there are no worlds inaccessible from i, then it may be that there are remoter and remoter accessible worlds without end, or it may be that some of the accessible worlds are maximally remote from i.

We may now give the truth conditions for the 'would' counterfactual in terms of a comparative similarity system, as follows.

> $\phi \mathbin{\Box\!\!\rightarrow} \psi$ is true at a world i (according to a given comparative similarity system) if and only if either
> (1) no ϕ-world belongs to S_i (the vacuous case), or
> (2) there is a ϕ-world k in S_i such that, for any world j, if $j \leqslant_i k$ then $\phi \supset \psi$ holds at j.

The counterfactual is true at i if and only if, if there is an antecedent-world accessible from i, then the consequent holds at every antecedent-world at least as close to i as a certain accessible antecedent-world.

The present formulation is exactly equivalent to the original formulation by means of spheres, without any restrictive assumptions. Recalling the way in which systems of spheres are supposed to carry information about comparative similarity, it is easily seen that we can put systems of spheres in one-to-one correspondence with comparative similarity systems, in such a way that the corresponding systems agree on the truth value at every world of every counterfactual. Starting with a comparative similarity system that assigns to each world i the relation $\leqslant_i$ and the set S_i, let $ be the assignment to each world i of the set $\$_i$ containing all and only those subsets S of S_i such that, whenever j belongs to S and k does not, $j <_i k$. Then it is easy to show (1) that $ is a system of spheres, and (2) that a counterfactual is true at a world according to the defined system of spheres $ if and only if it is true at that world according to the original comparative similarity system. Call $ the system of spheres *derived from* the original comparative similarity system. To go the other way, suppose we start with a system of spheres $. For each world i, let $j \leqslant_i k$ if and only if every sphere S in $\$_i$ that contains k also contains j; and let S_i be $\bigcup \$_i$. Then it is easy to show (1) that the assignment to each world i of the relation $\leqslant_i$ and the set S_i so defined is a comparative similarity system, and (2) that a counterfactual is true at a world according to this defined comparative similarity system if and only if it is true at that world according to the original system of spheres $. Say that this comparative similarity system is *derived from* the system of spheres $. We can show, finally, that for any comparative similarity system and system of spheres, the latter is derived from the former if and only if the former is derived from the latter.

The assignment to each world i of the sphere of accessibility S_i is the accessibility assignment corresponding to the outer necessity operator. It seems clumsy to assign the two separate items $\leqslant_i$ and S_i to each world i, but S_i is independent of $\leqslant_i$. If there are no $\leqslant_i$-maximal worlds, we know that S_i must be the whole set of worlds; but if there are some $\leqslant_i$-maximal worlds, we do not know from $\leqslant_i$ alone whether these are inaccessible worlds, to be left out of consideration in determining whether a counterfactual is true at i, or maximally remote accessible worlds. An alternative method* would be to let $\leqslant_i$ be an ordering not of all worlds, but only of accessible worlds, so that S_i could be defined as the field of the relation $\leqslant_i$; but this method is even clumsier.

There is something to be said for a philosophic conscience untroubled by possible worlds, but troubled by sets. After all, possible worlds have not led into paradox. The owner of such a conscience should prefer the present formulation to the original formulation involving an assignment to each world of a set of sets of worlds. He should regard a comparative similarity system, however, not as an assignment to each world of a two-place comparative similarity relation and a set of worlds regarded as accessible, but rather as a single three-place comparative similarity relation and a single two-place accessibility rèlation; or better still, as the two predicates '____ *is at least as similar to - - - as . . . is*' and '____ *is accessible from . . .*'.

2.4 Similarity Measures

I have sometimes spoken informally of *degrees* of similarity, as if similarity of worlds could be measured numerically; but I have not assumed that it could be. I have not used any quantitative concept of similarity, but only a comparative concept. One world is more similar than another to a third; but we need never say how much more, and the question how much more need not make sense.

Suppose, however, that we did have a quantitative concept of the similarity of worlds, so that we could speak sensibly of the degree of similarity, measured numerically, of one world to another. Then the truth conditions of 'would' counterfactuals would be as follows: $\phi \mathbin{\Box\!\!\rightarrow} \psi$ is true at a world i if and only if either (1) no ϕ-world is similar to i to a degree greater than zero, or (2) for some positive number d, there are ϕ-worlds similar to i to degree at least d, and ψ holds at every ϕ-world similar to i to degree at least d. (Worlds too unlike i to be considered—those that we previously regarded as lying outside all the spheres around i—are now assigned zero degree of similarity to i.)

* Followed in my 'Completeness and Decidability of Three Logics of Counterfactual Conditionals', *Theoria* 37 (1971): 74–85.

What additional assumptions do we make about comparative similarity orderings if we assume that they can be obtained from a numerical measure of similarity?

For one thing, we limit the number of gradations of similarity to the number of numbers, and we limit the order type of the comparative similarity ordering to the order types of orderings of numbers. Every similarity ordering with only countably many distinct gradations of similarity can be represented as derived from a numerical measure. Not every similarity ordering with more than countably many distinct gradations can be so represented; and no ordering with more distinct gradations than there are real numbers can be. This limitation hardly seems serious.

If we measure similarity numerically, and make uninhibited use of the analogy of similarity 'distance' between worlds to spatial distance between places, we are liable to make a much more serious and questionable assumption: that the degree of similarity of *i* to *j* equals the degree of similarity of *j* to *i*.* This assumption of symmetry for the similarity measure implies a constraint on similarity orderings derived from that measure: if $j <_i k$ and $k <_j i$, then $j <_k i$. But that constraint would be unjustified if we suppose that the facts about a world *i* help to determine which respects of similarity and dissimilarity are important in comparing other worlds in respect of similarity to the world *i*. The colors of things are moderately important at our world, so similarities and dissimilarities in respect of color contribute with moderate weight to the similarity or dissimilarity of other worlds to ours. But there are worlds where colors are much more important than they are at ours; for instance, worlds where the colors of things figure in fundamental physical laws. There are other worlds where colors are much less important than they are at ours; for instance, worlds where the colors of things are random and constantly changing. Similarities or dissimilarities in color will contribute with more or less weight to the similarity or dissimilarity of a world to one of those worlds where color is more important or less important. Thus it can happen that *j* is more similar than *k* to *i* in the respects of comparison that are important at *i*; *k* is more similar than *i* to *j* in the respects of comparison that are important at *j*; yet *i* is more similar than *j* to *k* in the respects of comparison that are important at *k*.

This assumption of symmetry is, of course, not an inevitable consequence of assuming that similarity of worlds admits of numerical measurement. We could have an asymmetric similarity measure. It

* Sobel, in 'Utilitarianisms: Simple and General', formulates essentially my analysis of counterfactuals by means of a numerical similarity measure, and does make this assumption.

would give us the degree of similarity of j to i, from the standpoint of i; that might not equal the degree of similarity of i to j, from the standpoint of j, because the relative importances of respects of comparison might differ from the standpoints of the two different worlds. (Having gone that far, we might as well have a function of three arguments that gives the degree of similarity of i to j from the standpoint of k, whether or not k is the same as i or j.) But why bother? The appeal of a numerical similarity measure comes from the analogy between similarity 'distance' and spatial distance. To the extent that the analogy breaks down, the point of having a numerical measure is lost.

2.5 Comparative Possibility

Ordinarily we think of possibility as an all-or-nothing matter. Something is possible or it is not, and the only way for it to be more possible that ϕ than that ψ is for it to be possible that ϕ but not possible that ψ. Given the notion of comparative overall similarity of worlds, however, there is a natural comparative concept of possibility. It is more possible for a dog to talk than for a stone to talk, since some worlds with talking dogs are more like our world than is any world with talking stones. It is more possible for a stone to talk than for eighteen to be a prime number, however, since stones do talk at some worlds far from ours, but presumably eighteen is not a prime number at any world at all, no matter how remote.

We may introduce into our language three comparative possibility operators:

$$\preccurlyeq$$

read as '*It is at least as possible that* ____ *as it is that* . . .', or as '*It is no more far-fetched that* ____ *than that* . . .', or as '*It is no more remote from actuality that* ____ *than that* . . .';

$$\prec$$

read as '*It is more possible that* ____ *than that* . . .', or as '*It is less far-fetched* (*less remote from actuality*) *that* ____ than *that* . . .'; and

$$\approx$$

read as '*It is equally possible* (*equally far-fetched, equally remote from actuality*) *that* ____ *and that* . . .'. The pair of $\preccurlyeq$ and $\prec$ are interdefinable, and $\approx$ is definable from $\preccurlyeq$, as follows:

$$\phi \preccurlyeq \psi =^{\text{df}} {\sim}(\psi \prec \phi),$$
$$\phi \prec \psi =^{\text{df}} {\sim}(\psi \preccurlyeq \phi),$$
$$\phi \approx \psi =^{\text{df}} \phi \preccurlyeq \psi \ \&\ \psi \preccurlyeq \phi.$$

Let us take $\preccurlyeq$ as primitive and the other two as defined. The truth conditions for $\preccurlyeq$ are given thus:

> $\phi \preccurlyeq \psi$ is true at a world i (according to a system of spheres $\$$) if and only if, for every sphere S in $\$_i$, if S contains any ψ-world then S contains a ϕ-world.

We thence obtain derived truth conditions for the defined operators $\prec$ and $\approx$:

> $\phi \prec \psi$ is true at i (according to $\$$) if and only if some sphere in $\$_i$ contains a ϕ-world but no ψ-world.
>
> $\phi \approx \psi$ is true at i (according to $\$$) if and only if all and only those spheres in $\$_i$ that contain ϕ-worlds contain ψ-worlds.

The outer and inner modalities, previously defined from the counterfactual, may now be redefined in terms of comparative possibility. The following definitions give the same derived truth conditions as the original ones. For the outer modalities:

$$\Diamond\phi =^{\mathrm{df}} \phi \prec \bot,$$
$$\Box\phi =^{\mathrm{df}} \sim\Diamond\sim\phi \quad \text{(or, directly, } \bot \approx \sim\phi\text{)}.$$

For any ϕ, $\phi \preccurlyeq \bot$ is everywhere true; thus the everywhere-false sentence $\bot$ is minimally possible. Outer possibility, then, is more-than-minimal possibility—possibility at least exceeding that of $\bot$. For the inner modalities:

$$\Diamond\!\!\!\!\cdot\;\,\phi =^{\mathrm{df}} \phi \approx \top,$$
$$\boxdot\phi =^{\mathrm{df}} \sim\Diamond\!\!\!\!\cdot\;\,\sim\phi \quad \text{(or, directly, } \top \prec \sim\phi\text{)}.$$

For any ϕ, $\top \preccurlyeq \phi$ is everywhere true; thus the everywhere-true sentence $\top$ is maximally possible. Inner possibility, then, is maximal possibility—possibility equal to that of $\top$. In a centered system of spheres, all and only truths are maximally possible; as we noted before, the inner modalities are trivial in this case.

Now we can reintroduce our counterfactual operators by definition from comparative possibility (and outer possibility, defined in turn from comparative possibility). The following definitions give the correct truth conditions.

$$\phi \Box\!\!\longrightarrow \psi =^{\mathrm{df}} (\phi \;\&\; \psi) \prec (\phi \;\&\; \sim\psi),$$
$$\phi \Diamond\!\!\longrightarrow \psi =^{\mathrm{df}} \sim(\phi \Box\!\!\longrightarrow \sim\psi) \quad \text{(or, directly, } (\phi \;\&\; \psi) \preccurlyeq (\phi \;\&\; \sim\psi)\text{)},$$
$$\phi \Box\!\!\rightarrow \psi =^{\mathrm{df}} \Diamond\phi \supset \phi \Box\!\!\longrightarrow \psi,$$
$$\phi \Diamond\!\!\rightarrow \psi =^{\mathrm{df}} \sim(\phi \Box\!\!\rightarrow \sim\psi) \quad \text{(or, } \Diamond\phi \;\&\; \phi \Diamond\!\!\longrightarrow \psi\text{)}.$$

We can just as well go the other way. Taking $\Box\!\!\rightarrow$ again as primitive, and defining $\Diamond\!\!\Rightarrow$ and $\Box\!\!\Rightarrow$ from it as in Section 1.6 (or taking one of the latter as primitive), we can introduce comparative possibility, with the correct derived truth conditions, by either of the following definitions. If we want to introduce $\preccurlyeq$ first,

$$\phi \preccurlyeq \psi =^{\text{df}} (\phi \vee \psi) \Diamond\!\!\Rightarrow \phi;$$

if we prefer to introduce $\prec$ first,

$$\phi \prec \psi =^{\text{df}} (\phi \vee \psi) \Box\!\!\Rightarrow \sim\psi.$$

We now have six alternative primitives: all of $\Box\!\!\rightarrow$, $\Diamond\!\!\rightarrow$, $\Box\!\!\Rightarrow$, $\Diamond\!\!\Rightarrow$, $\preccurlyeq$, and $\prec$ can be defined from any one of them.

So far, I have only been introducing new operators into the language to be interpreted. Truth conditions for sentences with this new vocabulary have been given by means of the same system of spheres already used to give truth conditions for counterfactuals. When we consider taking $\preccurlyeq$ rather than $\Box\!\!\rightarrow$ as our primitive, however, it becomes convenient to give truth conditions not by means of the system of spheres but by means of relations of comparative possibility among propositions (sets of worlds). Let us write

$$P \preccurlyeq_i Q$$

to mean that the proposition P is at least as possible, at the world i, as the proposition Q; let us write

$$P \prec_i Q \quad \text{(defined as: it is not the case that } Q \preccurlyeq_i P)$$

to mean that P is more possible at i than Q; and let us write

$$P \approx_i Q \quad \text{(defined as: } P \preccurlyeq_i Q \text{ and } Q \preccurlyeq_i P)$$

to mean that P and Q are equally possible at i. The symbols $\preccurlyeq$, $\prec$, and $\approx$ thus lead a double life, but there is no danger of confusion: unsubscripted, they are sentential connectives of the object language; subscripted, they are terms of the metalanguage denoting relations between propositions.

We may posit an assignment to each world i of a two-place relation $\preccurlyeq_i$ among propositions, regarded as the ordering of propositions in respect of their comparative possibility from the standpoint of the world i. Call such an assignment a *(centered) comparative possibility system* if and only if, for each world i, the following five conditions hold.

(1) The relation $\preccurlyeq_i$ is transitive; that is, whenever $P \preccurlyeq_i Q$ and $Q \preccurlyeq_i R$, then $P \preccurlyeq_i R$.

(2) The relation $\preccurlyeq_i$ is strongly connected; that is, for any propositions P and Q, either $P \preccurlyeq_i Q$ or $Q \preccurlyeq_i P$. (Equivalently: if $P \prec_i Q$ then $P \preccurlyeq_i Q$.)

(3) All and only truths are maximally possible; that is, the world i itself belongs to a proposition P if and only if, for every proposition Q, $P \preccurlyeq_i Q$. (Equivalently: if i belongs both to a proposition P and to a proposition Q, then $P \approx_i Q$; if i belongs to P but not to Q, then $P \prec_i Q$.)

(4) The union of a set of propositions is the greatest lower bound of the set. Let $\mathfrak{S}$ be a set of propositions and let $\bigcup\mathfrak{S}$ be the proposition containing all and only the worlds contained in members of the set $\mathfrak{S}$. Then $Q \preccurlyeq_i P$ for every P in $\mathfrak{S}$ if and only if $Q \preccurlyeq_i \bigcup\mathfrak{S}$. (If $\mathfrak{S}$ is finite, this means that $\bigcup\mathfrak{S}$ is as possible as the most possible member of $\mathfrak{S}$.)

(5) A singleton proposition that is more possible than every member of a set of propositions is also more possible than the union of the set. Let $\mathfrak{S}$ and $\bigcup\mathfrak{S}$ be as in (4) and let $\mathfrak{S}$ be nonempty; then if $\{j\} \prec_i P$ for every P in $\mathfrak{S}$, $\{j\} \prec_i \bigcup\mathfrak{S}$.

A comparative possibility system thus assigns to each world i a weak ordering of all propositions, with the propositions true at i itself—that is, containing i—together at the bottom. It follows from (4) that whenever Q is a subset of P, $P \preccurlyeq_i Q$. From that it follows further that for any proposition P, $P \preccurlyeq_i \Lambda$, where Λ is the empty set—in other words, the proposition true at no world, expressed by $\bot$ or any contradiction. As we shall see, the five conditions are required by the connection we intend between comparative possibility of propositions and comparative similarity of worlds.

It is easy to give the truth conditions for comparative possibility sentences according to a given comparative possibility system:

$\phi \preccurlyeq \psi$ is true at i if and only if $[\![\phi]\!] \preccurlyeq_i [\![\psi]\!]$.

(Recall that for any sentence ϕ, $[\![\phi]\!]$ is the set of worlds where ϕ is true, or in other words the proposition expressed by ϕ.) The derived truth conditions for $\prec$ and $\approx$ are similar:

$\phi \prec \psi$ is true at i if and only if $[\![\phi]\!] \prec_i [\![\psi]\!]$,
$\phi \approx \psi$ is true at i if and only if $[\![\phi]\!] \approx_i [\![\psi]\!]$.

The derived truth conditions for modal sentences and counterfactuals can simply be read off from the definitions of these in terms of the connectives of comparative possibility. For instance:

$\Diamond\phi$ is true at i if and only if $[\![\phi]\!] \prec_i \Lambda$.
$\phi \mathrel{\Box\!\!\rightarrow} \psi$ is true at i if and only if, if $[\![\phi]\!] \prec_i \Lambda$,
then $[\![\phi]\!] \cap [\![\psi]\!] \prec_i [\![\phi]\!] - [\![\psi]\!]$.

The present formulation by means of comparative possibility is exactly equivalent to the original formulation by means of spheres, with no restrictive assumptions. We can put systems of spheres in one-to-one correspondence with comparative possibility systems, in such a way that the corresponding systems agree on the truth value at every world of all counterfactuals and comparative possibility sentences. Starting with a comparative possibility system that assigns to each world i the relation $\preccurlyeq_i$, let \$ be the assignment to each world i of the set $\$_i$ of all and only those sets S of worlds such that, whenever a proposition P overlaps S and a proposition Q does not, then $P \prec_i Q$. Then it is easy to show (1) that \$ is a system of spheres, and (2) that a counterfactual or comparative possibility sentence is true at a world according to the defined system of spheres \$ if and only if it is true at that world according to the original comparative possibility system. Call \$ the system of spheres *derived from* the comparative possibility system. Starting rather with a system of spheres \$, let $P \preccurlyeq_i Q$ if and only if every sphere S in $\$_i$ that overlaps Q also overlaps P. Then it is easy to show (1) that the assignment to each world i of the relation $\preccurlyeq_i$ so defined is a comparative possibility system, and (2) that a counterfactual or comparative possibility sentence is true at a world according to this defined comparative possibility system if and only if it is true at that world according to the original system of spheres \$. Say that the comparative possibility system is *derived from* the system of spheres. Then we can show that for any comparative possibility system and system of spheres, the latter is derived from the former if and only if the former is derived from the latter.

Since comparative possibility systems and comparative similarity systems both can be put into one-to-one truth-preserving correspondence with systems of spheres, it follows that they can also be put into one-to-one truth-preserving correspondence with each other. This correspondence is quite simple: a comparative similarity ordering of worlds is, essentially, the corresponding comparative possibility ordering of singleton propositions, and accessible worlds are worlds that are more than minimally possible. More precisely: if the comparative similarity system that assigns to each world i the similarity ordering $\leqslant_i$ of worlds and the sphere of accessibility S_i corresponds to the comparative possibility system that assigns to each world i the ordering $\leqslant_i$ of propositions, then (1) $j \leqslant_i k$ if and only if $\{j\} \preccurlyeq_i \{k\}$, and (2) j belongs to S_i if and only if $\{j\} \prec_i \Lambda$.

2.6 *Cotenability*

In order to compare my theory with most previous theories of counterfactuals (this will be done in Sections 3.1–3.3) it will be helpful to reformulate my truth conditions in terms of relations of logical implication and of 'cotenability' between sentences.

Let us say that χ is *cotenable* with ϕ at a world i (according to a system of spheres $) if and only if either (1) χ holds throughout $\bigcup \$_i$, or (2) χ holds throughout some ϕ-permitting sphere in $\$_i$. In other words: if and only if either (1) χ holds at all worlds accessible from i, or (2) some ϕ-world is closer to i than any $\sim\chi$-world. A necessary truth (in the sense of outer necessity) is cotenable with anything; a falsehood is cotenable with nothing; between these limits, cotenability is a matter of comparative possibility. If ϕ is entertainable at i, χ is cotenable with ϕ at i if and only if $[\![\phi]\!] \prec_i [\![\sim\chi]\!]$.

A counterfactual $\phi \mathbin{\Box\!\!\rightarrow} \psi$ is true at i (according to my truth conditions) if and only if the premise ϕ and some auxiliary premise χ, cotenable with ϕ at i, logically imply ψ. Proof: Suppose there is some such premise χ. Perhaps there is no ϕ-permitting sphere around i, in which case $\phi \mathbin{\Box\!\!\rightarrow} \psi$ is vacuously true at i. Otherwise there is a ϕ-permitting sphere throughout which χ holds; since ϕ and χ jointly imply ψ, $\phi \supset \psi$ also holds throughout this sphere; so $\phi \mathbin{\Box\!\!\rightarrow} \psi$ is true. Conversely, suppose $\phi \mathbin{\Box\!\!\rightarrow} \psi$ is true at i. Either there is no ϕ-permitting sphere around i, in which case $\sim\phi$ is a premise, cotenable with ϕ at i, which together with ϕ implies ψ; or else there is a ϕ-permitting sphere throughout which $\phi \supset \psi$ holds, in which case $\phi \supset \psi$ is a premise, cotenable with ϕ at i, which together with ϕ implies ψ. Q.E.D.

If each of $\chi_1, \ldots, \chi_n$ is cotenable with ϕ, then so is their conjunction; so we can also say that $\phi \mathbin{\Box\!\!\rightarrow} \psi$ is true at i if and only if ϕ together with finitely many premises $\chi_1, \ldots, \chi_n$, each cotenable with ϕ at i, logically imply ψ.

That would be the customary way to give truth conditions by means of cotenability, but there is an easier way: $\phi \mathbin{\Box\!\!\rightarrow} \psi$ is vacuously true at i if and only if $\sim\phi$ is cotenable with ϕ at i, non-vacuously true at i if and only if $\phi \supset \psi$ is cotenable with ϕ at i.

2.7 *Selection Functions*

The simplest and most direct formulation of the idea that a counterfactual is true if and only if the consequent holds at the closest antecedent-worlds depends, unfortunately, on the objectionable Limit Assumption. Suppose we are given a system of spheres $ that satisfies the Assumption (for antecedents in our language). That means, we recall, that for every world i and antecedent ϕ (in our language) that is entertainable at i, there is a smallest ϕ-permitting sphere around i. The ϕ-worlds in that sphere are the closest ϕ-worlds to i. If an antecedent ϕ is not entertainable at i, then the set of closest ϕ-worlds to i is empty.

We may define a function f which selects, for any sentence ϕ and world i, the set $f(\phi, i)$ of closest ϕ-worlds to i. Let

$$f(\phi, i) = \begin{cases} \text{the set of } \phi\text{-worlds belonging to every } \phi\text{-permitting sphere in } \$_i, \text{ if there is any } \phi\text{-permitting sphere in } \$_i, \\ \text{the empty set otherwise.} \end{cases}$$

We call f a *selection function* (or a *set-selection function*, when we wish to compare these with Stalnaker's world-selection functions to be considered in Section 3.4). We will say that f is *derived from* the given system of spheres $\$$. We have already seen how to give the truth conditions for counterfactuals under the Limit Assumption. In terms of the selection function: the 'would' counterfactual $\phi \mathrel{\Box\!\!\rightarrow} \psi$ is true at a world i if and only if ψ is true at every world in $f(\phi, i)$. Similarly the 'might' counterfactual $\phi \mathrel{\Diamond\!\!\rightarrow} \psi$ is true at i if and only if ψ is true at some world in $f(\phi, i)$.*

When f is derived from a system of spheres $\$$ that satisfies the Limit Assumption, then these truth conditions will agree with my original truth conditions given in terms of the system of spheres. But if $\$$ does not satisfy the Limit Assumption, then there will be disagreement. Sometimes $f(\phi, i)$, as I have defined it, will be empty because, although there are ϕ-permitting spheres around i, yet there is no ϕ-world that belongs to every ϕ-permitting sphere. Then $\phi \mathrel{\Box\!\!\rightarrow} \psi$ will incorrectly come out as vacuously true at i for any consequent ψ, contrary to my original truth conditions in terms of the system of spheres.

We may call a function f from sentences and worlds to sets of worlds a *(centered) (set-) selection function* if and only if, for all sentences ϕ and ψ and for each world i, the following four conditions hold.

(1) If ϕ is true at i, then $f(\phi, i)$ is the set $\{i\}$ having i as its only member.
(2) $f(\phi, i)$ is included in $[\![\phi]\!]$.
(3) If $[\![\phi]\!]$ is included in $[\![\psi]\!]$ and $f(\phi, i)$ is nonempty, then $f(\psi, i)$ also is nonempty.
(4) If $[\![\phi]\!]$ is included in $[\![\psi]\!]$ and $[\![\phi]\!]$ overlaps $f(\psi, i)$, then $f(\phi, i)$ is the intersection of $[\![\phi]\!]$ and $f(\psi, i)$.

* The use of set-selection functions in this way to give an analysis of counterfactuals has been suggested by John Vickers, and further studied by Peter Woodruff in 'Notes on Conditional Logic' (duplicated, May 1969, University of California at Irvine).

Such selection functions turn out to be all and only the selection functions derived from (centered) systems of spheres satisfying the Limit Assumption. If f is derived from some such system of spheres, then it is easily verified that conditions (1)–(4) are satisfied. Conversely, suppose f satisfies conditions (1)–(4). Let $ be the assignment to each world i of the set $\$_i$ of all and only those sets S of worlds such that, first, every world in S belongs to some $f(\phi, i)$; and, second, whenever $[\![\phi]\!]$ overlaps S, $f(\phi, i)$ is included in S. Then it is easily verified that $ is a system of spheres satisfying the Limit Assumption. It remains to show that f is derived from $. Proof: first consider the case that there is no ϕ-permitting sphere in $\$_i$. The union of the sets $f(\psi, i)$ for all sentences ψ—including ϕ—is obviously a sphere in $\$_i$; if this sphere is not ϕ-permitting although it includes $f(\phi, i)$, then $f(\phi, i)$ must be empty, so in this case $f(\phi, i)$ agrees with the selection function derived from $. Next consider the case that there is some ϕ-permitting sphere in $\$_i$. By (2) and the definition of $, $f(\phi, i)$ is included in the intersection of $[\![\phi]\!]$ and any ϕ-permitting sphere; so to show that in this case also $f(\phi, i)$ agrees with the selection function derived from $, it suffices to find a sphere S in $\$_i$ such that the intersection of $[\![\phi]\!]$ and S is exactly $f(\phi, i)$. Take S to be the union of the sets $f(\psi, i)$, for all sentences ψ such that $[\![\phi]\!]$ is included in $[\![\psi]\!]$. Whenever $[\![\chi]\!]$ overlaps S, $[\![\chi]\!]$ overlaps some $f(\psi, i)$ such that $[\![\phi]\!]$ is included in $[\![\psi]\!]$. $[\![\phi]\!]$ is included also in $[\![\chi \vee \psi]\!]$, so $f(\chi \vee \psi, i)$ is included in S. $[\![\chi]\!]$ overlaps $f(\chi \vee \psi, i)$; if not, it would follow by (2)–(4) that $f(\psi, i)$ and $f(\chi \vee \psi, i)$ are the same, contradicting the fact that $[\![\chi]\!]$ does overlap $f(\psi, i)$. It now follows by (4) that $f(\chi, i)$ is included in $f(\chi \vee \psi, i)$, and hence in S; so S is a sphere. The intersection of $[\![\phi]\!]$ and S is the union of all intersections of $[\![\phi]\!]$ with a set $f(\psi, i)$ such that $[\![\phi]\!]$ is included in $[\![\psi]\!]$. By (4), each such intersection is either the empty set or $f(\phi, i)$; and by (2), one such intersection—that of $[\![\phi]\!]$ with $f(\phi, i)$—is $f(\phi, i)$; so the intersection of $[\![\phi]\!]$ and the sphere S is exactly $f(\phi, i)$, as desired. Q.E.D.

The truth-preserving correspondence between systems of spheres satisfying the Limit Assumption and the selection functions derived from them is not, in general, one-to-one. Given a system of spheres $ satisfying the Limit Assumption, we may be able to add a new sphere S around some world i in such a way that S does not become the smallest ϕ-permitting sphere around i for any sentence ϕ. Let $' be the new system of spheres obtained by the addition of S; then $' also satisfies the Limit Assumption, and $ and $' yield the same derived selection function. This means that systems of spheres sometimes carry more information about comparative similarity than is needed to determine the truth values at all worlds of all counterfactuals. The same is true of comparative similarity and comparative possibility systems since these, as we have seen, stand in one-to-one truth-preserving correspondence with systems of spheres.

My conditions (2)–(4) imply that if $[\![\phi]\!]$ and $[\![\psi]\!]$ are the same—if ϕ and ψ are different sentences expressing the same proposition—then $f(\phi, i)$ and $f(\psi, i)$ are the same. Therefore instead of taking the first arguments of selection functions to be sentences, as I did, I could just as well have taken them to be propositions expressible by sentences.

Instead of taking a single function f of two arguments, I could just as well have taken an assignment to each world i of a function f_i of one argument—a sentence or a proposition, according to taste. Instead of $f(\phi, i)$ we then have $f_i(\phi)$ or $f_i([\![\phi]\!])$. If we take the arguments to be propositions rather than sentences, and if we let $\mathcal{U}_i$ be the set of expressible propositions A such that $f_i(A)$ is nonempty, then each $\langle \mathcal{U}_i, f_i \rangle$ is what Bengt Hansson has called a *choice structure*.*

We might also reformulate a selection function as a family, indexed by sentences or by propositions, of assignments to worlds of spheres of accessibility; or, more traditionally, as a family, indexed by sentences or by propositions, of accessibility relations between worlds. A world j is *accessible*$_\phi$ (or *accessible*$_{[\phi]}$ if we prefer) from a world i if and only if j is in $f(\phi, i)$. For any fixed antecedent ϕ, we can regard $\phi \Box\!\!\rightarrow$ and $\phi \Diamond\!\!\rightarrow$ as if they were a pair of one-place modal operators, interpreted as usual by means of accessibility$_\phi$. $\phi \Box\!\!\rightarrow \psi$ is true at i if and only if ψ holds at every world accessible$_\phi$ from i; $\phi \Diamond\!\!\rightarrow \psi$ is true at i if and only if ψ holds at some world accessible$_\phi$ from i. My analysis of variably strict conditionals, as restricted by the Limit Assumption, can thus be subsumed as a special case under a general theory of sententially or propositionally indexed modalities.‡ My analysis in its full generality, on the other hand, cannot be thus subsumed—the Limit Assumption is essential.

* 'Choice Structures and Preference Relations', *Synthese* **18** (1968): 443–458. We saw that every selection function is derived from a system of spheres; this could have been obtained as a corollary of Hansson's theorem that whenever $\langle \mathcal{U}, f \rangle$ is a choice structure such that $\mathcal{U}$ is closed under finite unions, there is a weak ordering R *underlying* $\langle \mathcal{U}, f \rangle$—that is, for each A in $\mathcal{U}$, $f(A)$ is the set of members of A that bear R to every member of A. In this case, the ordering underlying $\langle \mathcal{U}_i, f_i \rangle$ is the comparative similarity ordering $\leqslant_i$.

‡ A general account of propositionally indexed modalities is to be found in Brian F. Chellas, 'Basic Conditional Logic', *Journal of Philosophical Logic* **4** (1975): 133–153.

2.8 The Selection Operator

If we tolerate the Limit Assumption and use a selection function to interpret the counterfactual, we can express the selection function by an operator in the object language. We can define the counterfactual connectives from that operator, plus other logical apparatus.

Introduce a one-place sentential operator $\not{f}$, called the *selection operator*. We may read it as '*Things are the way they would be if it were the case that* ____'. The sentence $\not{f}\phi$ is to be true at all and only the selected, closest ϕ-worlds given by our selection function f. The counterfactual may (provisionally) be defined thus:

$$\phi \mathbin{\Box\!\!\rightarrow} \psi =^{\text{df}} \Box(\not{f}\phi \supset \psi).$$

This definition makes the counterfactual into a strict conditional after all; however the antecedent ϕ of the counterfactual is not the same as the antecedent $\not{f}\phi$ of the strict conditional. According to this definition the counterfactual is true if and only if ψ holds at all $\not{f}\phi$-worlds—that is, at all of the selected, closest ϕ-worlds.

That is more or less what we want, but the account so far is incomplete. The $\not{f}\phi$-worlds are the 'selected, closest ϕ-worlds'. Selected from, and closest to, what world? A selection function has *two* arguments. I said that $\not{f}\phi$ was to be true at all and only the worlds in $f(\phi, i)$—but without specifying which world is i.

Lennart Åqvist, in first proposing the use of a selection operator, specified that the selection was to be done always from the standpoint of our actual world.* That is, $\not{f}\phi$ was to be true at all and only the closest ϕ-worlds to ours—all and only the worlds in $f(\phi, i)$ where i is our actual world. Since Åqvist selects always from the standpoint of a single world, ours, his selection functions have no need of a second argument.

That will do so long as we care only to say how the *actual* truth values of counterfactuals depend on the truth values at various worlds of their antecedents and consequents. But we have been more ambitious hitherto. Counterfactuals are, for the most part, contingent. We have been trying to give their truth conditions in general: to say how the truth value of a counterfactual at *any* world depends on the truth values at various worlds of its antecedent and consequent. Even if we are ultimately interested only in the actual truth values of sentences, still we must consider the truth values of counterfactuals at other worlds than ours to obtain the actual truth values of sentences in

* 'Modal Logic with Subjunctive Conditionals and Dispositional Predicates', *Journal of Philosophical Logic* 2 (1973): 1–76.

which counterfactuals are embedded inside other counterfactuals. For instance, Åqvist's method cannot be made to account for the truth of the sentence

(*I look in my pocket* $\Box\!\!\rightarrow$ *I find a penny*) & (*There is no penny in my pocket* $\Box\!\!\rightarrow$ $\sim$(*I look in my pocket* $\Box\!\!\rightarrow$ *I find a penny*)).

In order to make proper use of the selection operator, we must recognize that there are sentences that cannot naturally be assessed simply for truth or falsehood at a world, but rather call for a three-place truth relation: truth *of* a sentence ϕ *at* a world *i with reference to* a world *j*. For instance, '*Things are better*' is true at *i* with reference to *j* if and only if things are better at *i* than at *j*, but there is no natural way to assess '*Things are better*' for truth at a world without some reference world to serve as a standard of comparison. Similarly, we cannot give a satisfactory general account of the truth conditions for the selection operator by means simply of the two-place truth-at relation. We should rather use the three-place truth relation and say that $\not{f}\phi$ is true at a world *i* with reference to a world *j* if and only if *i* belongs to $f(\phi, j)$—that is, if and only if *i* is a closest ϕ-world to *j*.

What shall we do when we need to mix these special sentences that require the three-place truth relation with ordinary sentences that can be treated adequately by means of the two-place truth relation? Although it is unsatisfactory to treat special sentences as though they were ordinary, it is harmless to treat ordinary sentences as though they were special. Let us call an ordinary sentence true at *i* with reference to any *j* if and only if it is true at *i*. Given this stipulation, we can explain the difference between ordinary and special sentences by saying that an ordinary sentence is true at a world with reference to all worlds or none, whereas a special sentence is sometimes true at a world with reference to some worlds but not others.

Just as the truth conditions for ordinary sentences are formulated in terms of the two-place truth relation, so parallel truth conditions for special sentences—or for ordinary sentences treated as special in order to compound them with special sentences—may be given in terms of the three-place truth relation. The reference world tags along throughout. For instance, a material conditional is true at *i* with reference to *j* if and only if either the consequent is true at *i* with reference to *j* or the antecedent is false at *i* with reference to *j*. A sentence $\Box\phi$ is true at *i* with reference to *j* if and only if ϕ is true with reference to *j* at every world accessible from *i*; the appropriate accessibility assignment for outer modality is given in terms of the selection function by specifying that a world is accessible from *i* if and only if it belongs to some $f(\chi, i)$ or other.

Suppose ϕ and ψ are ordinary sentences. Then the strict conditional $\Box(f\phi \supset \psi)$ that we took provisionally to define $\phi \Box\!\!\rightarrow \psi$ is in many cases a special sentence. It is true at a world *i* with reference to a world *j* if and only if $f\phi \supset \psi$ is true with reference to *j* at every world accessible from *i*; that is, if and only if ψ is true with reference to *j* at every world accessible from *i* where $f\phi$ is true with reference to *j*; that is, if and only if ψ is true at every world in $f(\phi, j)$ that is accessible from *i*. Taking *j* as *i*, the accessibility restriction becomes redundant. Thus $\Box(f\phi \supset \psi)$ is true at *i* with reference to *i* itself if and only if ψ is true at every world in $f(\phi, i)$; that is, if and only if $\phi \Box\!\!\rightarrow \psi$ is true at *i*.

That is good enough to explain our success with actual truth values, but it is not quite right. We want to define $\phi \Box\!\!\rightarrow \psi$ by an ordinary sentence that will be true at a world *i* if and only if our provisional definiens $\Box(f\phi \supset \psi)$ is true at *i* with reference to *i*. We must provide a new operator † with truth conditions as follows: $\dagger\chi$ is an ordinary sentence true at *i* if and only if χ is true at *i* with reference to *i*. (When χ is ordinary, $\dagger\chi$ has the same truth conditions as χ itself.) Prefixing the †-operator to our provisional definens, we obtain the correct definition.* With it we have a parallel definition for the 'might' counterfactual.

$$\phi \Box\!\!\rightarrow \psi =^{\text{df}} \dagger\Box(f\phi \supset \psi),$$
$$\phi \Diamond\!\!\rightarrow \psi =^{\text{df}} \dagger\Diamond(f\phi \ \& \ \psi).$$

We could go instead in the other direction and define the selection operator from the counterfactual, using propositional quantification and another special operator for the three-place truth relation. Let $\downarrow\chi$ be true at *i* with reference to *j* if and only if χ is true at *j* with reference to *j*; a sentence $\downarrow\chi$ will in most cases be special. Now we may define f thus:

$$f\phi =^{\text{df}} \forall\xi(\downarrow(\phi \Box\!\!\rightarrow \xi) \supset \xi).$$

(Here ξ is to be any propositional variable that does not occur in ϕ.) Intuitively, the definiens says that whatever *would* hold, if ϕ did, *does* hold.‡

* Correct on the assumption that the antecedent and consequent are ordinary. But we might want to drop that assumption. For instance, if we want to handle countercomparatives like '*If my yacht were longer, things would be better*' without quantifying in, we will want to take the antecedent and consequent as special sentences. But if we want to use the object-language selection operator in the presence of special antecedents and consequents, then we must consider extra-special sentences that require a four-place truth relation—and so on up.

‡ The ↓-operator and the †-operator, or rather their temporal analogs, were first introduced by Hans Kamp and Frank Vlach, respectively. They are needed for symbolizing such sentences as '*Jones is going to remember* (*simultaneously*) *everyone now living*' and '*Jones was once going to remember* (*simultaneously*)

everyone then living' in a language without overt quantification over times. 'Now' or 'then' is ↓, 'once' is †. See Kamp, 'Formal Properties of "Now"', *Theoria* 37 (1971): 227–273; and Vlach, '"Now" and "Then": a Formal Study in the Logic of Tense and Anaphora' (Ph.D. dissertation, 1973, University of California at Los Angeles). Åqvist, in an appendix to 'Modal Logic with Subjunctive Conditionals and Dispositional Predicates' (written subsequently to the body of the work) has adopted my suggestion to use the †-operator along with a selection operator that yields special rather than ordinary sentences.