Handout for Phil 4300

Definition: A gamble on A made at odds of a:b for stakes of $S is such that it pays $bS/(a+b) if A is true and –$aS/(a+b) if A is false. (i.e. if you are betting with a bookie, you pay $a to place a bet where you would win back $(a+b) and net $b in winnings if you win the bet.) Since the stakes are arbitrary, odds of a:b are the same as odds of ax : bx for any x.

Definition: A conditional gamble on A given B at odds of a:b at stakes $S is such that it pays $bS/(a+b) if A&B is true, pays –$aS/(a+b) if (~A)&B, and pays 0 if ~B (i.e. the bet is called off).

Definition: An agent considers a gamble fair if the subjective expected utility of the gamble is $0.

Definition: An agent's betting quotient (or Degree of Belief) for a proposition A [abbreviated DoB(A)]is the value q such that the agent considers a bet at price $q for stake $1 to be fair.

-- Assuming that the agent values money linearly, [so that U($x) + U($y) = U($x+y)], this means that if the agent considers a bet on A at odds a:b fair, then the agent's betting quotient is a/(a+b).

Unconditional Betting
Variation on the example from class today (Mon, March 4th):

Imagine that you post betting quotients of .3 for P, .4 for Q, and .6 for P∨Q and that P and Q are exclusive (cannot both be true at the same time). The third axiom of probability says that P(p) + P(q) = P(p∨q) when p and q are exclusive so your betting quotients violate this axiom. That means that you can be Dutch Booked. Here is how to construct such a book. The idea is that the quotients on P and Q are 'too high' and the quotient for P∨Q is 'too low'. There is nothing objective about 'too high' or 'too low' here, just relative to each other for internal consistency. So you want to make the agent bet so that she wins some bets and loses some, but will be guaranteed to lose more than she wins overall. To do this, make her win if the things she thinks are 'too high' are true and lose when the things she thinks are 'too low' are true. So in this case, make three bets all with stakes $1 and make her bet in favor of P, in favor of Q, and against P∨Q.

So here are the three bets all of which she considers fair:
Bet on P such that she wins .7 if P is true and pays out .3 if P is false.
Bet on Q such that she wins .6 if Q is true and pays out .4 if Q is false.
Bet against P∨Q such that she wins .6 if P∨Q is false and pays out .4 if P∨Q is true.

Now since P and Q are mutually exclusive, there are three possible outcomes:

P happens, Q does not. So P∨Q is also true. The agent wins .7 (P), and pays out .4 (Q) and .4 (P∨Q) and so nets -.1

P does not happen but Q does. So P∨Q is also true. The agent wins .6 (Q), and pays out .3 (P) and .4 (P∨Q) and so nets -.1

Neither P nor Q happens. The agent wins .6 (against P∨Q) but pays out .3 (P) and .4 (Q) and so nets -.1

Thus the agent loses money no matter what.

In the case we did in class, P(p∨q) was .8 so you should reverse all the bets so that the agent loses if P, loses if Q, and wins if P∨Q. If the stakes are all $1, then again the agent will net -.1 no matter what.

In class the conclusion was that the agent lost .5 no matter what. I am not sure what I did wrong, but that is wrong!

The agent's probability for P∨Q is too high so make her win if P∨Q and lose if P and lose if Q.

Here are the three cases:
P & ~Q the agent loses .7 (P), wins .4 (~Q), and wins .2 (P∨Q)
~P & Q the agent wins .3 (P), loses .6 (Q), and wins .2 (P∨Q)
~P&~Q the agent wins .3 (P), wins .4 (Q), and loses .8 (P∨Q)

Conditional betting:

Lets call DoB(A&B) = q
Lets call DoB(B) = r
For simplicity and clarity, we can assume here DoB(~B) = 1-r

A conditional bet on A given B at price $DoB(A&B)/DoB(B) to win $1 has the following payoff table:

	Total payoff in dollars = winnings minus amount paid for bet
A & B	1-(q/r)
A & ~B	0
~A & B	-q/r
~A & ~B	0

We can simulate a conditional bet on A given B by making two unconditional bets – one on A&B and one on ~B.

In order to come up with the two bets, we aim to have the same payoff table as above. We are going to make a bet on A&B and a bet on ~B. The easiest way to make sure we have the right payoff table is to aim to have 0 total payoff in the ~B cases. Since we will lose the A&B bet, we make sure that the total net winnings on the ~B bet exactly cancel out the cost of placing the A&B bet.

The following theorem makes this easier to see how to do:
Theorem: $P(A|B) = P(A\&B) + P(\sim B) \times P(A\&B)/P(B)$

Thus if we wanted to simulate a bet on A given B with stakes of \$1, we could use a bet on A&B stakes of \$1 and a bet on ~B at stakes of \$DoB(A&B)/DoB(B) = \$q/r

Bet 1: Bet on A&B at \$q to win \$1 – this is odds q:1-q
Bet 2: Bet on ~B at \$(q/r) x (1-r) to win \$q/r – this is odds (q/r) x (1-r) : (q/r) – (q/r) x (1-r) which is the same as odds (1-r) : r

Now the results of making these two bets together:

	Payoff bet 1	Payoff bet 2	Total Payoff
A & B	1-q	-(q/r) x (1-r)	1-(q/r)
A & ~B	-q	(q/r) – [(q/r) x (1-r)] = q	0
~A & B	-q	-(q/r) x (1-r)	-q/r
~A & ~B	-q	(q/r) – [(q/r) x (1-r)] = q	0

Notice that the payoff table for making both of these bets is exactly the same as making a single conditional bet. Therefore, if you value the single conditional bet at a different rate than you value the sum of these two bets, you can be Dutch Booked.