Philosophy 4300 – Decision Theory
Spring 2019
Example from Apr 8, 2019

If we allow mixed strategies, then every finite game has at least one equilibrium. This was first proved by John Nash of "A Beautiful Mind" fame. Resnik proves only that this is true of 2 player 2x2 zero-sum games. It is not too difficult to find the equilibrium in such a game. The basic idea is to imagine that the equilibrium pair is (p, q) where p = Prob that row plays R1 and q = Prob that column plays C1. Then since Row's payoff will be a weighted average of EU(R1) and EU(R2), if the payoff for playing R1 with probability p is ≥ EU(R1) and also is ≥ EU(R2) then it must be that EU(R1) = EU(R2). This generalizes to more than two strategies. The probability distribution for column's moves must be such that it makes all of row's moves equally good and the probability distribution for row's moves must be such that column's moves are now all equally good.

The example we used in class of a 3x3 game was a modified rock, paper, scissors game.

| | Rock | Paper | Scissors |
|---|---|---|---|
| Rock | 0,0 | -1,1 | 2,-1 |
| Paper | 1,-1 | 0,0 | -1,2 |
| Scissors | -1,1 | 1,-1 | 0,0 |

Here, we get the same win/loss/tie results, but row gets a little extra bonus if they win with rock and column gets a little extra bonus if they win with scissors. Intuitively, this should have some effect on the 'normal' game's equilibrium of 1/3, 1/3, 1/3.

Let's first figure out how often column will play the three different strategies at equilibrium. In class I had three variables – P(Rc) = Rc, P(Pc) = Pc, P(Sc)=Sc. That is certainly possible. To save a few steps, I will just use the fact that Rc+Pc+Sc=1 when constructing the EU equations by replacing Sc with 1-Rc-Pc.

EU(Row plays R) = 0(Rc) -1(Pc) + 2(1-Rc-Pc) = 2 – 2Rc – 3Pc
EU(Row plays P) = 1(Rc) +0(Pc) -1(1-Rc-Pc) = 2Rc + Pc -1
EU(Row plays S) = -1(Rc) +1(Pc) +0(1-Rc-Pc) = Pc – Rc

The $1^{st}$ = $2^{nd}$ so 2 – 2Rc – 3Pc = 2Rc + Pc -1 so therefore 4Rc + 4Pc = 3
The $1^{st}$ = $3^{rd}$ so 2 – 2Rc – 3Pc = Pc – Rc so therefore Rc + 4Pc = 2
Take this first result minus the second: 3Rc = 1 so Rc = 1/3
Substitute back in and 4/3 + 4Pc = 3 so 4Pc = 5/3 so Pc = 5/12
Since they add to 1, Sc = 3/12 = 1/4. If you plug each of these back into the equations, you will see that the Expected Utility of any move for row (and so also for any mixed strategy) is 1/12.

To find the probability of row's moves, find the values where the expected utility of columns moves are all equal.

Here we say that P(row plays R) = Rr, P(row plays P) = Pr, and P(row plays S) = 1-Rr-Pr.

EU(Col plays R) = 0(Rr) -1(Pr) + 1(1-Rr-Pr) = 1 – 1Rr – 2Pr
EU(Col plays P) = 1(Rr) +0(Pr) -1(1-Rr-Pr) = -1 +2Rr + 1Pr
EU(Col plays S) = -1(Rr) +2(Pr) +0(1-Rr-Pr) = 0 -1Rr + 2Pr

The 1$^{st}$ = 2$^{nd}$ so 1 – 1Rr – 2Pr = -1 +2Rr + 1Pr so 3Rr + 3Pr = 2
The 1$^{st}$ = 3$^{rd}$ so 1 – 1Rr – 2Pr = 0 -1Rr + 2Pr so 4Pr = 1
So Pr = 1/4, Rr = 5/12 and so Sr = 4/12 = 1/3. Plugging these values into the expected utility calculations will again yield that each of column's moves yield 1/12 on average.

Thus the equilibrium pair for this game is Row plays the following mixed strategy:

Row: P(Rock) = 5/12, P(Paper) = 1/4, P(Scissors) = 1/3
Column: P(Rock) = 1/3, P(Paper) = 5/12, P(Scissors) = 1/4