Math 309 — Introductory Lecture

New York state offers a lottery game called Quick-Draw. In one version of the game, a player chooses four numbers from 1 to 80. The state then chooses twenty numbers from 1 to 80. A new game is played every four minutes. The number of matches determines the payout.

Number of matches	0	1	2	3	4
Payout on a $1 bet	0	0	1	5	55

Question: *What is the probability of each outcome?*

After the player chooses four numbers, there are 4 "good" numbers and 76 "bad" numbers. The number of "good" items in a sample without replacement from a finite population is a *random variable* from a *hypergeometric distribution*. In this case, the probability that exactly x matches occur (there are x "good" items in the sample) is given by the formula

$$p(x) = \frac{\binom{4}{x}\binom{76}{20-x}}{\binom{80}{20}}$$

Numerically, the distribution is:

x	0	1	2	3	4
$p(x)$	0.3083	0.4327	0.2126	0.0432	0.0031

Question: *If one played many times, what would the average return be?*

In a large number of plays betting $1, you would expect to have 0 or 1 match about 74.1% of the time and lose the $1 bet, to have 2 matches about 21.3% of the time and get the $1 back, to have 3 matches about 4.3% of the time and get $5, and to match all 4 about 0.3% of the time to win $55. The average return would likely be close to

$$(0.7410)(0) + (.2126)(1) + (.0433)(5) + (.0031)(55) \doteq 0.597$$

In other words, you would expect to lose about 40 cents per dollar bet on average. While the average return has some chance of being either higher or lower than this value, the *law of large numbers* implies that if an individual made a sufficiently large number of bets, the average return would almost certainly be close to this value.

In November of 1997, New York state offered a promotion where payouts on this game were doubled on Wednesdays.

Number of matches	0	1	2	3	4
Payout on a $1 bet	0	0	2	10	110

Question: *What is the expected return for the game with doubled payouts?*

The average return would be twice as large, or $1.194. This is *larger* than the amount of the $1 bet. In other words, you would expect to profit by a bit more than 19 cents on average for each dollar bet. If one could bet a large number of times, the probability of a profit would become almost certain!

Question: *How could one take advantage of this unusual betting opportunity?*

The answer is to bet as frequently as possible, with the largest feasible dollar amount on each bet. Clearly, to do this, you would need to be able to use winnings on earlier bets for subsequent bets.

It was possible to make twenty individual bets with a single game card. Also, it was possible for each individual bet to be for 1, 2, 5, or 10 dollars. Any tickets with payouts less than $600 could be converted to cash easily. Larger payouts needed to be cashed at a special regional center. Because the top prize for a $5 bet is $550, the best strategy is to make as many $5 bets as possible. A single action of wagering twenty separate $5 bets would cost $100, and could be repeated a couple thousand times in a day.

Question: *What does the distribution of profit look like for twenty $5 bets?*

The most likely outcome would be not winning the large payout with any of the twenty bets. The probability of this is about $(0.997)^{20} \doteq 0.9405$. The chance of winning the large payout exactly once in twenty tries is about $20(0.003)(0.997)^{19} \doteq 0.0578$, and the chance of winning twice or more is about 0.0017. The random number of large payouts in a fixed number of independent individual bets, each with the same probability of winning, is described by the *binomial distribution*. The distribution of profit contains a large clump with a mean profit a little below 0, a small clump with a mean profit a little below near $550, and minuscule clumps at higher profits.

Question: *How many bets would it take on average until the first win of a large payout?*

The random variable that waits until the first success in a sequence of independent trials with the same success probability is called the *geometric distribution*. The probability of a large payout, calculated with more accuracy, is about 1/326. If about one payout in 326 is big, you expect to wait about 326 bets between each big payout. However, the distribution of big payouts won't be that regular. Some waits will be much shorter, while occasionally, the waits will be substantially longer.

Question: *If someone began with a stake of $2000, what would the distribution of profit look like after 400 $5 bets?*

The distribution of profit will be centered at $5n \times 400 \times 0.194 \doteq 389$ dollars, but there is still considerable spread. While it is more likely than not that a profit will have been made, there is still about a one in four chance that the net gain after 400 bets will be negative. However, it is quite unlikely that more than $1000 of the original stake is lost. The distribution, although still rather skewed, is beginning to look like a symmetric unimodal bell-shaped normal curve.

Question: *How many bets are necessary for the probability of a negative profit to be less than 1%?*

We assume the number to be large and the distribution of profits to be approximately *normal*. This is a result of *the central limit theorem*. A normal curve is described by its *mean* (centering point) and *standard deviation* (a measure of spread interpreted as a typical deviation from the mean). The standard deviation is the square root of the *variance*, which is the average squared distance from the mean. The mean and standard deviation of the profit for a single $5 bet are $0.973 and $31.94 respectively. The mean and standard deviation of the profit distribution after n $5 bets are $n(0.973)$ and $\sqrt{n}(31.94)$ respectively. The probability that there is a negative profit after n bets is approximately

$$\Phi\left(\frac{0 - n(0.974)}{\sqrt{n}(31.94)}\right) = \Phi\big(-\sqrt{n}(0.0305)\big)$$

where Φ is the standard normal *cumulative distribution function*. This goes to 0 as $n \to \infty$ and is approximately 0.01 when $n = 5826$.

So, with a stake of a few thousand dollars, it is possible to realize a positive profit before going broke. Even if the profit is not positive, it is quite likely that some part of the initial stake remains. Ascertaining the probability of going broke given a specific large initial stake is not an easy calculation.

Question: *Suppose it were feasible to make 2400 betting actions (resulting in 48,000 individual bets) in a single day. What would the distribution of profit look like after this many $5 bets?*

Now the distribution is well described by a normal distribution with a mean of $46,734 and a standard deviation of $6,998.

True Story! According to a professor friend of mine whom I trust, two recent graduates of another university who had taken a probability course in college recognized that New York had made an error and run a promotion that could allow them to make a profit as I have described. New York ran the promotion in November of 1997 for three consecutive Wednesdays and the students realized profits of over $120,000.

Taking this course does not guarantee that you will be so lucky to win thousands of dollars in a state sponsored lottery. You will, however, gain the skills to identify any potential similar opportunities. (New York State clearly made a mistake. The current promotion gives a free play if you buy ten, which, as we have seen, is worth about 60 cents on average, and the state would gain about four dollars before giving the promotion.)

More practically, you will be able to ascertain the distribution of profit after playing the usual gambling games and find that the probability of *losing* great deals of money is overwhelming when playing standard state sponsored lotteries where the expected profit on any single play is negative. Knowing how to judge probabilities of gambling games accurately can help you avoid the playing games where you could easily thousands of dollars in a year.