An Objective Justification of Bayesianism II: The Consequences of Minimizing Inaccuracy

Hannes Leitgeb and Richard Pettigrew

December 19, 2009

Abstract

One of the fundamental problems of epistemology is to say when the evidence in an agent's possession justifies the beliefs she holds. In this paper and its prequel, we defend the Bayesian solution to this problem by appealing to the following fundamental norm:

> Accuracy An epistemic agent ought to minimize the inaccuracy of her partial beliefs.

In the prequel, we made this norm mathematically precise; in this paper, we derive its consequences. We show that the two core tenets of Bayesianism follow from the norm, while the characteristic claim of the Objectivist Bayesian follows from the norm along with an extra assumption. Finally, we consider Richard Jeffrey's proposed generalization of conditionalization. We show not only that his rule cannot be derived from the norm, unless the requirement of Rigidity is imposed from the start, but further that the norm reveals it to be illegitimate. We end by deriving an alternative updating rule for those cases in which Jeffrey's is usually supposed to apply.

1 Introduction

It is often said that the epistemic norms governing full beliefs are justified by the more fundamental epistemic norm, *Try to believe truths*. For instance, the synchronic norm that demands that an agent have a consistent set of full beliefs at any given time follows from this along with the fact that the propositions in an inconsistent set of beliefs cannot possibly all be true together. Similarly, the diachronic norm that demands that an agent update her beliefs by valid rules of inference follows from this fundamental norm along with the fact that a valid rule of inference preserves truth from premises to conclusion.

In this paper, we attempt to justify the Bayesian's putative norms governing partial beliefs in a similar way. We will appeal to the more fundamental norm, *Approximate the truth*, which is plausibly the analogue of the fundamental norm for full beliefs stated above. From this, we will derive the central tenets of Bayesianism; we will show that the characteristic claim of the Objectivist Bayesian also follows from this norm in the presence of a further, rather strong assumption; and we will cast doubt on one of the other extensions to Bayesianism proposed in the literature.

However, before we begin, we must present the framework of partial beliefs, the Bayesian norms, and the precise version of the accuracy norm stated above. The derivation of this precise version was the subject of this paper's prequel ("An Objective Justification of Bayesianism I: Measuring Inaccuracy"), and we presuppose the conclusion of that prequel in what follows.

In this paper, as in its prequel, we will be concerned only with agents who have an opinion about only a finite set of possible worlds. As in the prequel, if W is such a set of possible worlds, let $\mathcal{P}(W)$ denote the power set of W, and let $\mathrm{Bel}(W)$ denote the set of functions $b : \mathcal{P}(W) \to \mathbb{R}_0^+$. The functions in $\mathrm{Bel}(W)$ are (potential) *belief functions* on the power set of W. It is a presupposition of any form of Bayesianism that, if W is the set of possible worlds about which an agent holds an opinion, then that agent's epistemic state at a given time t may be represented quantitatively, by a belief function $b_t \in \mathrm{Bel}(W)$ that takes each proposition A, represented as a subset of W, to a real number $b_t(A)$ that measures the degree of credence the agent assigns to A.

The first tenet of Bayesianism is a synchronic norm. Indeed, it is the analogue of the synchronic norm for full beliefs stated above: *An agent ought to have a consistent set of beliefs.* The Bayesian demands that an agent has a *coherent* belief function.

> **Probabilism** For any time, t, an agent's belief function b_t at time t ought to be a probability measure on the power set of W: that is, (i) for all $A \subseteq W$, $b_t(A) \geq 0$; (ii) $b_t(\varnothing) = 0$ and $b_t(W) = 1$; and (iii) for any disjoint $A, B \subseteq W$, $b_t(A \cup B) = b_t(A) + b_t(B)$.[1]

The second tenet is diachronic and might be thought of as analogous to a diachronic norm for full beliefs that demands that an agent updates by applying valid rules of inference.[2] It is characteristic of virtually all forms of Bayesianism (at least as long only plain factual evidence about the world is concerned):

> **Conditionalization** Suppose that, between t and t', an agent learns proposition $E \subseteq W$ with certainty and nothing more. And suppose further that $b_t(E) \neq 0$.[3] Then her belief function $b_{t'}$ at time t' ought to be such that, for each $A \subseteq W$,
>
> $$b_{t'}(A) = b_t(A|E) =_{df.} \frac{b_t(A \cap E)}{b_t(E)}$$

Together, **Probabilism** and **Conditionalization** constitute the core of Bayesianism. Various philosophers have added various further claims, but they have not gained the unanimous support of the faithful. Two such further proposals will be of particular interest to us here.

The first is the characteristic claim of the Objectivist Bayesian.[4] In our context, in which the agent has an opinion about only finitely many possible worlds, this amounts to the following norm:

[1] Bayesians are divided on whether to demand also that belief functions satisfy countable additivity in those cases in which W is infinite; see [25] for a general criticism of requiring countable additivity. In this paper, we consider only the case in which W, and thus its power set, is finite. Thus, this question will not arise.

[2] Of course, not all philosophers agree that there is such a norm for full beliefs. See, for instance, [8] and [12].

[3] **Conditionalization** prescribes an updated belief function only when the piece of evidence learned was not completely ruled out by the agent's original belief function: that is, **Conditionalization** says nothing of how an agent with belief function b ought to update her belief function upon receiving evidence E where $b(E) = 0$. The norms that govern such situations are interesting, and we will have a little more to say about them when we consider Jeffrey's proposed extension of the core Bayesian tenets. However, a full epistemic account of these cases would require an extension of our theory to a more general class of belief functions, as e.g. Popper functions (cf. [23]), which we leave as an open problem.

[4] We use the term 'Objectivist Bayesianism' to mean the conjunction of **Probabilism**, **Uniform Distribution**, and **Conditionalization**. Often it is used to cover the conjunction of **Probabilism** and **Conditionalization** with any principle that specifies a rational prior belief function for an agent. However, while these proposals sometimes differ in those cases in which W is infinite, they rarely deviate from **Uniform Distribution** when W

Uniform Distribution Suppose W is finite. And suppose that, at time t, E is the strongest proposition given to the agent by her evidence. Then her belief function b_t at t ought to be such that, for all $A \subseteq W$,

$$b_t(A) = \frac{|A \cap E|}{|E|}$$

In particular, if the agent has not learned any evidence by t, then $E = W$, and her belief function ought to be such that, for all $A \subseteq W$,

$$b_t(A) = \frac{|A|}{|W|}$$

Bayesians who do not subscribe to **Uniform Distribution** are known as *subjectivists*: see, for instance, Part II of [29]. Having given our argument for **Probabilism** and **Conditionalization** in sections 6.1 and 6.2, we give an argument for **Uniform Distribution** in section 6.3. However, it relies on rather a strong assumption, which may be read as begging the question. Thus, we present it much more tentatively than the others. Our main interest in it consists in observing which additional assumptions one might make in order to extend our justification of Bayesianism simpliciter to one of its Objectivist variants.

The second proposed extension of Bayesianism that will concern us here was advanced by Richard Jeffrey. So far, we have assumed on behalf of the Bayesian that an agent acquires new evidence only when she learns the truth of a particular proposition with certainty. Jeffrey denied this: he claimed that new evidence can take a form different from the one considered in **Conditionalization**. Moreover, he argued for a rule that specified how an agent should respond to this different sort of evidence: see chapter 11, [15]. Here is his rule:[5]

Jeffrey Conditionalization Suppose $\{E_1, \dots, E_m\}$ is a partition of W, $0 \leq q_1, \dots, q_m$, and $q_1 + \cdots + q_m = 1$. Suppose that, between t and t', the agent obtains evidence that imposes the following side constraints on belief function $b_{t'}$: for $i = 1, \dots, m$,

$$b_{t'}(E_i) = q_i$$

Then, she ought to have a belief function $b_{t'}$ at t such that, for each $A \subseteq W$,

$$b_{t'}(A) = \sum_{i=1}^{m} q_i \cdot b_t(A|E_i)$$

is finite. Thus, our terminology is quite standard. See, for instance, [1], [14], and [16]. See [32] for an overview of ways in which an agent's belief function can be objective while additionally being constrained by certain kinds of empirical knowledge. We also want to stress that the term 'Objective' as used in the title of our paper is meant to characterize the manner of justification that we after, which should not be confused with the target of justifying Objectivist Bayesianism. Indeed, we are mainly interested in defending *subjective* Bayesianism in this article.

[5] As Carl Wagner pointed out to us, Jeffrey did not actually propose his updating rule in the form given here. In the form he proposed, extra side constraints are placed on the updated belief function $b_{t'}$. In particular, Jeffrey requires *Rigidity* with respect to all partition sets E_i: that is, for all $A \subseteq W$ and all E_i, $b_{t'}(A|E_i) = b_t(A|E_i)$. It is easy to show that Jeffrey's rule is in fact equivalent to, or uniquely determined by, these extra constraints on $b_{t'}$. So once these side constraints are subsumed under the overall constraints that the target belief function at time t' has to satisfy, then there is no room for discussion anymore on what the right method of updating is in such a situation. In the light of this, we will concentrate on the version of Jeffrey's epistemic norm that is stated in **Jeffrey Conditionalization**, which has interested many philosophers, e.g. [28], independently of how Jeffrey introduced the rule originally. For more on the philosophical status and justification of Rigidity, see [3]. We will return to the topic of Rigidity in section 7.5.

providing $b_t(E_i) \neq 0$ for all $i = 1, \ldots, m$.

In section 7, we will show not only that one cannot extend the justification that we will give of **Probabilism** and **Conditionalization** in order to justify **Jeffrey Conditionalization**; we will show further that **Jeffrey Conditionalization** is illegitimate in certain circumstances, since it does not always minimize inaccuracy. Fortunately, an alternative method of update is available which respects, and is commanded by, inaccuracy minimization, and in the last part of this paper we study its properties.

2 Our justification of Bayesianism: the outline

So much for the Bayesian norms; let's turn to our attempt to justify them. We will present this attempt in outline here, then survey and critique other attempts, and then return to our justification to fill in the details.

In the prequel to this paper, we argued for a particular way of making the following norm precise:

> Accuracy An agent ought to approximate the truth. In other words: she ought to minimize her inaccuracy.

We began by introducing the notions of (potential) *local* and *global inaccuracy measures*. A local inaccuracy measure is a mathematical function that takes a proposition A, a world w, and a real number $x \in \mathbb{R}_0^+$ to a measure $I(A, w, x)$ of the inaccuracy of the degree of belief x in proposition A at world w. And a global inaccuracy measure is a function that takes a belief function b and a possible world w to a measure $G(w, b)$ of the inaccuracy of b at w.

With these definitions in hand, we introduced the notions of *expected* local and global inaccuracy. The *expected local inaccuracy of degree of belief x in proposition A by the lights of belief function b, with respect to local inaccuracy measure I, and over the set E of epistemically possible worlds* is defined as follows:

$$\mathrm{LExp}_b(I, A, E, x) = \sum_{w \in E} b(\{w\}) I(A, w, x)$$

While the *expected global inaccuracy of belief function b' by the lights of belief function b, with respect to global inaccuracy measure G, and over the set E of epistemically possible worlds*, is defined similarly:

$$\mathrm{GExp}_b(G, E, b') = \sum_{w \in E} b(\{w\}) G(w, b')$$

Using these notions, we argued for the following four more precise versions of Accuracy. First, the two synchronic versions:

> Accuracy (Synchronic expected local) An agent ought to minimize the expected local inaccuracy of her degrees of credence in all propositions $A \subseteq W$ by the lights of her *current* belief function, relative to a *legitimate* local inaccuracy measure, and over the set of worlds that are *currently* epistemically possible for her.

> Accuracy (Synchronic expected global) An agent ought to minimize the expected global inaccuracy of her current belief function by the lights of her *current* belief function, relative to a *legitimate* global inaccuracy measure, and over the set of worlds that are *currently* epistemically possible for her.

The latter condition is close in spirit Allan Gibbard's 'minimal requirement' that an agent ought to have a belief function that is *immodest* relative to a measure of inaccuracy ([9]). Like Gibbard, we appeal to the obvious norm that one ought not to have a belief function that is worse *by its own lights* than it needs to be.

Second, the two diachronic versions of the Accuracy norm, where an agent has learned evidence between t and t' that imposes constraints C on her belief function $b_{t'}$ at time t', or on the set E of worlds that are epistemically possible for her at t', or both:

> Accuracy (Diachronic expected local) *At time* t', such an agent ought to have a belief function that satisfies constraints C and is minimal amongst belief functions thus constrained with respect to the expected local inaccuracy of the degrees of credence it assigns to each proposition $A \subseteq W$ by the lights of her belief function *at time* t, relative to a *legitimate* local inaccuracy measure, and over the set of worlds that are epistemically possible for her *at time* t' given the constraints C.
>
> Accuracy (Diachronic expected global) *At time* t', such an agent ought to have a belief function that satisfies constraints C and is minimal amongst belief functions thus constrained with respect to expected global inaccuracy by the lights of her belief function *at time* t, relative to a *legitimate* global inaccuracy measure, and over the set of worlds that are epistemically possible for her *at time* t' given the constraints C.

To complete our specification of these mathematically precise versions of Accuracy, we required a characterization of the legitimate inaccuracy measures, both local and global. To obtain this, we showed that the only measures that do not lead any agent who follows these norms into three different undesirable epistemic dilemmas are the *quadratic inaccuracy measures*. That is, the legitimate local inaccuracy measures are those of the form:

$$I(A, w, x) = \lambda(\chi_A(w) - x)^2$$

where $\chi_A : W \to \{0, 1\}$ is the characteristic function of the set A. And the legitimate global inaccuracy measures are those of the form:

$$G(w, b) = \lambda||w - b_{\text{glo}}||^2$$

where w and b are represented by their corresponding vectors—that is, w_i is represented by the unit vector $(\delta_{i,1}, ..., \delta_{i,n})$ and b is represented by the vector that we call the global belief function $b_{\text{glo}} = (b(\{w_1\}), ..., b(\{w_n\}))$ to which b gives rise—and $||u - v||$ is the Euclidean distance between the vectors u and v, i.e. $||u - v|| = \sqrt{(u_1 - v_1)^2 + ... + (u_n - v_n)^2}$. These characterizations of the legitimate local and global inaccuracy measures are called Local and Global Inaccuracy Measures, respectively.

Note that, in the presence of Local and Global Inaccuracy Measures, and on the basis of our results on accuracy in the prequel to this paper, it is easy to show that the following implications hold:[6]

$$\text{Accuracy (Synchronic expected local)} \Rightarrow \text{Accuracy (Synchronic expected global)}$$
$$\text{Accuracy (Diachronic expected local)} \Rightarrow \text{Accuracy (Diachronic expected global)}$$

It is also easy to see that neither converse holds. After all, the global versions of the norm impose constraints only on the *global* belief function $b_{\text{glo}} = (b(\{w_1\}), ..., b(\{w_n\}))$ to which the belief function b gives rise. And there are many belief functions that give rise to the same global belief function. Thus, the global versions of the norms can impose no constraints on the values of $b(A)$ when A is not a singleton proposition $\{w_i\}$ with $w_i \in W$. So, even if the global versions of the Accuracy norm can be satisfied only by one *global* belief function $b_{\text{glo}} = (b(\{w_1\}), ..., b(\{w_n\}))$, they can nonetheless be satisfied by many different belief functions, where those belief function agree on the singleton propositions.

However, the global versions of the norms are far from idle; indeed, as we shall see, in one situation that we will consider, they are essential. In section 6.1, we will show that it follows from Accuracy (Synchronic expected local) that, at any time t, an agent's belief function b_t at t ought to be a probability function. Now, while there are many belief functions that give rise to a particular global belief function, there is only one probability function that gives rise to it. Thus, if the global versions of Accuracy demand a particular *global* belief function, then together with Accuracy (Synchronic expected local) they demand a particular belief function, namely, the unique probability function to which that global belief function gives rise.

It will turn out that exactly this sort of reasoning is demanded by our discussion of those instances of Accuracy (Diachronic expected local) and Accuracy (Diachronic expected global) that cover the cases with which **Jeffrey Conditionalization** is concerned—see section 7. For it turns out that while the relevant instances of the latter norm can always be satisfied, some of the relevant instances of the former cannot.[7] Thus, in these instances of Accuracy (Diachronic expected global), we must appeal to Accuracy (Synchronic expected local) in order to narrow the range of belief functions that the norm permits—as we will see, in these case, it has the effect of narrowing that range from many to one.

So much for the relations between the various versions of the Accuracy norm. Let us turn to their consequences. In this paper, we derive **Probabilism** from Accuracy (Synchronic expected local) (section 6.1) and **Conditionalization** from Accuracy (Diachronic expected local) (section 6.2), both on the assumption of Local Inaccuracy Measures. We derive **Uniform Distribution** from Accuracy (Synchronic expected local) and Local Inaccuracy Measures along with a rather strong extra assumption called Minimize (section 6.3). And, as we have noted above, if we assume Local Inaccuracy Measures, we find that the instances of Accuracy (Diachronic expected local) relevant to **Jeffrey Conditionalization** cannot always be satisfied; however, on the assumption of Global Inaccuracy Measures, the relevant instances of Accuracy (Diachronic expected global) can be satisfied. We show that Jeffrey's updating rule does not always satisfy it, and we describe the rule that does (section 7).

[6] To see this, note first that, if $I(A, w, x) = \lambda(\chi_A(w) - x)^2$ and $G(w, b) = \lambda||w - b_{\text{glo}}||^2$, the $\text{LExp}_b(I, \neg E, E, x)$ is minimal for $x = 0$. Then note that, if $b'(\neg E) = 0$, then

$$\text{GExp}_b(G, E, b') = \sum_{w \in E} \text{LExp}_b(I, \{w\}, E, b'(\{w\}))$$

which follows in exact analogy to the proof of theorem 3 of the prequel to this paper.

[7] We prove the latter part of this statement in section 9.3 in the appendix.

3 Other justifications of Bayesianism

Before we give this justification, let us compare our strategy to other putative justifications of the tenets of Bayesianism. The form of our argument is this: We identify a desirable property of belief functions—namely, minimal expected inaccuracy by the lights of the best available belief function—and we define this property with mathematical precision in Local and Global Inaccuracy Measures; then we show that an agent satisfies the norms that follow from the desirability of this property if, and only if, her belief function satisfies the constraints imposed by Bayesianism, which is the topic of this article. As Alan Hájek [11] points out, this is the form in which the most important arguments for **Probabilism** must be presented if they are to be valid: in the case of the synchronic Dutch Book argument of Ramsey [24] and de Finetti [4], the desirable property is *invulnerability to Dutch Book bets*; for van Fraassen [29], it is *calibration*; Ramsey's argument from his representation theorem turns on the normativity of a certain set of *rationality constraints*—see [24] again; and, for Joyce [17], **Probabilism** follows from the desirability of *gradational accuracy*.

The same observation holds for belief change and the most important arguments in favour of **Conditionalization**: Lewis' diachronic Dutch Book argument [21] relies on the same desirable feature as the synchronic version mentioned above; Lange [19] derives the Bayesian updating rule from the desirability of calibration; Williams' argument [31] is premised on the assumption that an agent's belief function ought to encode no more information than is available to him, where informational content is measured by Shannon's entropy measure; van Fraassen's symmetry argument [29] demands that an agent's updating rule assign to epistemically equivalent inputs epistemically equivalent outputs, deriving **Conditionalization** from these symmetry conditions; and, finally, Greaves and Wallace [10] show that **Conditionalization** follows from the normative claims of decision theory, if each property out of a class of properties of belief functions is considered desirable.

The single argument in favour of **Uniform Distribution** also fits the pattern that Hájek identifies: Jaynes' argument [14] for this tenet of Objectivist Bayesianism appeals to the desirability of a belief function with maximal Shannon entropy relative to the available evidence.

In the presence of these powerful arguments for Bayesianism, we must justify making our own attempt. We share with Joyce the conviction that the ultimate desideratum for a belief function is that it *be close to the truth*, i.e., that it have what one may call *gradational accuracy*. Now suppose an agent were presented with the option of gaining greater expected gradational accuracy at the cost of Dutch Book vulnerability, calibration, diachronic coherence in van Fraassen's sense, or Shannon entropy relative to her accumulated evidence. We submit that she should take that option. Though it is obvious that these other features are desirable, it is equally obvious that they are trumped by minimal expected inaccuracy as far as purely epistemic considerations are concerned. Despite the obvious joys and dangers of betting, and despite the practical consequences of disastrous betting outcomes, an agent would be irrational *qua* epistemic being if she were to value her invincibility to Dutch Books so greatly that she would not sacrifice it in favour of a belief function that she expects to be more accurate. And the same is true of the other features upon which the arguments enumerated above turn. For instance, we value Shannon entropy because it seems to measure the extent to which we have come to our opinion purely on the basis of the evidence; however, it would be irrational not to go beyond the evidence if in doing so one was aware of being guaranteed to decrease one's expected inaccuracy. And so on.

Thus, following Hájek's line of reasoning, we raise the following objection against all but the arguments of Joyce, and Greaves and Wallace. In each argument given, the tenets of Bayesianism are derived from some desideratum. However, in all cases except that of Joyce and Greaves and Wallace, the desideratum is not ultimate epistemologically: that is, there are epistemic desiderata that trump the desideratum to which the argument appeals. Thus, it simply does not follow from the corresponding argument that an agent ought to satisfy the constraints of Bayesianism, for nothing in the argument precludes a situation in which the desideratum to which the argument appeals is trumped by a more compelling desideratum and where satisfying this latter desideratum requires the agent to violate Bayesianism. Thus, the arguments given above are invalid, and can be made valid only by the introduction of an implausible premise, i.e., asserting that the desideratum in question is ultimate. Only when we derive Bayesianism from the ultimate epistemic desideratum of minimal expected inaccuracy—of closeness to the truth formalized in the context of partial beliefs—can we claim to have established it.

Before we turn to our own justification of the Bayesian tenets, we will consider briefly Joyce's argument for **Probabilism** and the argument of Greaves and Wallace in favour of **Conditionalization**.

4 Joyce's argument for **Probabilism**

In [17] and [18], Joyce puts forward what he calls a 'non-pragmatic' justification of **Probabilism**. This, he hopes, will replace the pragmatic justifications that are based on Dutch Book arguments and against which he raises powerful objections. In those papers, he employs a strategy very similar to the one that we shall employ here to establish all of the tenets of Bayesianism; indeed, [17] was a significant source of inspiration for the present paper. For instance, we share with Joyce the focus on accuracy as the central epistemic virtue. However, as will become apparent below, the detailed execution of this shared strategy differs in our case and in Joyce's. In particular, the notion of *expected* inaccuracy will be central to our argument, while it plays no part in Joyce's theory in [17] nor in the central theorem (Theorem 2) of [18]. Moreover, as we will see, we impose other conditions on inaccuracy than Joyce, and we use them to defend not just **Probabilism** but in fact we will deal with all of the tenets of Bayesianism.

In [17], Joyce presented six properties that a global inaccuracy measure must possess, and showed that, by the lights of any global inaccuracy measure with these properties, for every belief function b that violates **Probabilism**, there is a belief function b' that satisfies it, such that b' is more accurate than b at every possible world. Of course, this does not establish **Probabilism** unless it is also the case that there is no belief function b'' that violates **Probabilism** and which is at least as accurate than b' at all possible worlds. The six properties to which Joyce appeals in [17] do not guarantee this, but, in [18], he states four different properties that do guarantee both claims. If G is a global inaccuracy measure and b is a belief function, we say that *b is admissible relative to G* just in case there is *no* belief function b' such that $G(b', w) \leq G(b, w)$ for all $w \in W$ with strict inequality in at least one case. Then we can state Joyce's theorem as follows (Theorem 2 of [18]):

Theorem 1 (Joyce) *Suppose $G(w, b)$ is a global inaccuracy measure that satisfies the following four conditions:*

(1) Truth-Directedness *Suppose $b = (\alpha_1, ..., \alpha_n)$ and $b' = (\alpha'_1, ..., \alpha'_n)$ are belief functions*

and $w = (\delta_1, ..., \delta_n) \in W$. Then, if $|\alpha_i - \delta_i| \leq |\alpha'_i - \delta_i|$ for all $i = 1, ..., n$, with strict inequality for at least one i, then $G(w, b) < G(w, b')$.

(2) Coherent Admissibility *Each probability function is admissible relative to G.*

(3) Finitude $G(w, b) \in \mathbb{R}$ *for all b and w.*

(4) Continuity *For any world w, $G(w, \_)$ is a continuous function.*

Then,

(i) *Each non-probability function b is not admissible relative to G. Furthermore, there is a probability function b' such that $G(w, b') \leq G(w, b)$ for all $w \in W$ with strict inequality for at least one $w \in W$.*

(ii) *Each probability function b is admissible relative to G.*

Clearly, the controversial claim is Coherent Admissibility since it accords a privileged status to probability functions. We are inclined to ask: Why is it that we are justified in demanding that every probability function is admissible? Why are we not justified in demanding the same of a belief function that lies outside that class? And, of course, we *must not* make this demand of any non-probability function; if we do, (i) will not follow.

Joyce defends Coherent Admissibility as follows (p. 279, [18]). Prior to an argument for **Probabilism**, we are not justified in saying that the probability functions are the *only* rational belief functions, but we are justified in saying that they lie *amongst* the rational belief functions. After all, for any probability function b, it is at least possible that an agent obtain evidence that the objective chance of each $A \subseteq W$ is $b(A)$. Thus, if Lewis' Principal Principle is correct, we would not want a scoring rule that precludes this belief function as rational.

The problem with this argument is that it restricts the scope of Joyce's result. If this is the justification of Coherent Admissibility, then Joyce's argument for **Probabilism** will only apply to an agent with a belief function that can be realized as a possible representation of objective chances. And there are many agents with belief functions that cannot be realized in this way. Alan Hájek gives a nice example (p. 246-249, [11]). Suppose one of the propositions about which the agent has an opinion is *The chance of the next coin toss landing heads up is $\frac{1}{2}$*. Maybe this proposition either does not have an objective chance, or its objective chance is 0 or 1. But it is quite possible that the agent's evidence leads her, quite rationally, to assign degree of credence $\frac{1}{2}$ to that proposition. Since her resulting belief function is not guaranteed to be rational by appealing to objective chances and the Principal Principle, it does not fall within the scope of Coherent Admissibility and thus Joyce's argument does not establish that it should satisfy **Probabilism**. Furthermore: wouldn't it be problematic if a supposedly purely epistemological justification of Bayesianism relied on properties of chance and on probabilistic reflection principles relating credence and chance?

5 Greaves and Wallace's argument for Conditionalization

A global epistemic utility function U takes a belief function and a world to the epistemic utility of having that belief function at that world. In [10], Greaves and Wallace offer a justification for **Conditionalization** that turns on the following property of global epistemic utility functions (p. 625, [10]):

Weak Propriety Suppose U is a legitimate global epistemic utility function and that b_1 and b_2 are probability functions. Then, if $E \subseteq W$,

$$\sum_{w \in W} b_1(\{w\}|E)U(b_2, w) \leq \sum_{w \in W} b_1(\{w\}|E)U(b_1(\,\cdot\,|E), w)$$

Put informally, this says that, relative to a legitimate global epistemic utility function, updating in accordance with **Conditionalization** yields a belief function that does not expect that any other way of updating would have produced greater epistemic utility.

The putative justification for Weak Propriety seems to be analogous to Joyce's argument for Coherent Admissibility, though without the additional appeal to objective chances and the Principal Principle. Greaves and Wallace note that, prior to an argument for **Conditionalization**, we are not justified in believing that it is the *only* rational way to incorporate new evidence; but, they claim, we are justified in believing that it is *one* rational way. (Unlike Joyce, they do not give any argument for this claim.) Thus, we should rule out global epistemic utility functions on which **Conditionalization** yields a belief function that expects another updating rule to have produced greater epistemic utility.

Our objection to Weak Propriety is slightly different from our objection to Coherent Admissibility. As we saw in section 3, there are many epistemic virtues: e.g., accuracy, Dutch Book invulnerability, potential calibration, and so on. While we consider the updating rule **Conditionalization** to be rational, this may be because we judge that it preserves just one of these virtues; perhaps the virtue of potential calibration (cf. [19]). If this is the case, there is no reason to think that an epistemic utility function that aligns utility with accuracy will satisfy Weak Propriety even though such a utility function is clearly legitimate. Thus, the effect of Weak Propriety is to limit the class of legitimate utility functions to those that measure whatever epistemic virtues **Conditionalization** preserves. This might be regarded as begging the question.

Thus, although Greaves and Wallace's justification of **Conditionalization** is in several respects quite close to ours—as is Joyce's justification of **Probabilism**—we do not actually endorse it. Furthermore, even if we did, it would not be clear how it could be generalized to an argument for **Probabilism** and **Uniform Distribution**. It will be important to show that the technique by which we establish the synchronic tenet(s) of Bayesianism can also establish its diachronic tenet. We hope that our argument will have this advantage over the arguments of Joyce and Greaves and Wallace.

6 Our justification of Bayesianism: the argument in detail

Finally, we turn to a detailed presentation of our justification of Bayesianism. As promised, it depends on the synchronic and diachronic versions of the local and global versions of the Accuracy norm. In particular:

(1) **Probabilism** follows from Accuracy (Synchronic expected local) (section 6.1).

(2) **Conditionalization** follows from Accuracy (Diachronic expected local) (section 6.2).

(3) **Uniform Distribution** follows from a related, but stronger norm (section 6.3).

(4) We show that, in the situations usually supposed to be covered by **Jeffrey conditionalization**, there is no updating rule that satisfies Accuracy (Diachronic expected local)

(section 9.3). However, there is such a rule that satisfies (the strictly weaker norm) Accuracy (Diachronic expected global). We describe the rule that does this, and note that it is not Jeffrey's rule and that Jeffrey's rule in fact violates the norm in certain circumstances (section 7).

6.1 Probabilism and Accuracy (Synchronic expected local)

Suppose E is the set of worlds that are epistemically possible for an agent and suppose that I is a quadratic local inaccuracy measure. Then, by Accuracy (Synchronic expected local), her belief function must be such that, for every proposition A, the expected local inaccuracy of the degree of credence $b(A)$ in A by the lights of b, relative to I, and over the epistemically possible worlds in E is minimal. This entails **Probabilism** by the following theorem:

Theorem 2 *Suppose b is a belief function, $E \subseteq W$, $\sum_{w \in E} b(\{w\}) \neq 0$,[8] and I is a quadratic local inaccuracy measure. Then the following two propositions are equivalent:*

(i) *For all $A \subseteq W$ and any $x \in \mathbb{R}_0^+$,*

$$\mathrm{LExp}_b(I, A, E, b(A)) \leq \mathrm{LExp}_b(I, A, E, x)$$

(ii) *b is a probability function with $b(E) = 1$.*

The proof is given in the appendix (section 9.1.1).

6.2 Conditionalization and Accuracy (Diachronic expected local)

Suppose an agent has a belief function b_t at time t and suppose that I is a quadratic local inaccuracy measure. Suppose further that, between t and a later time t', she obtains evidence that restricts the set of worlds that are epistemically possible for her to the set $E \subseteq W$, where W is the set of epistemically possible worlds at t. Then, by Accuracy (Diachronic expected local), her new belief function $b_{t'}$ at t' must be such that, for every proposition A, the expected local inaccuracy of the degree of credence $b(A)$ in A by the lights of b_t, relative to I, and over the 'new' set E of epistemically possible worlds must be minimal. This entails **Conditionalization** by the following theorem:

Theorem 3 *Suppose b_t and $b_{t'}$ are probability functions, $E \subseteq W$, $\sum_{w \in E} b_t(\{w\}) \neq 0$, and I is a quadratic local inaccuracy measure. Then the following two propositions are equivalent:*

(i) *For all $A \subseteq W$ and any $x \in \mathbb{R}_0^+$,*

$$\mathrm{LExp}_{b_t}(I, A, E, b_{t'}(A)) \leq \mathrm{LExp}_{b_t}(I, A, E, x)$$

(ii) *For all $A \subseteq W$,*

$$b_{t'}(A) = \frac{b_t(A \cap E)}{b_t(E)} =_{df.} b_t(A|E)$$

As above, the proof is given in the appendix (section 9.1.2).

Note that, in this theorem, we presuppose that the belief functions in question are probability functions. This is permitted by the result of section 6.1 that an agent's belief function must be a probability function, on pain of epistemic irrationality.

[8] If $\sum_{w \in E} b(\{w\}) = 0$, then $\mathrm{LExp}_b(I, A, E, x) = 0$ for all x. So any choice of x would minimize $\mathrm{LExp}_b(I, A, E, x)$, though in a completely trivial way, which is why we exclude this case from the start.

6.3 **Uniform Distribution** and **Minimize**

Next, we consider **Uniform Distribution**, the distinctive claim of Objectivist Bayesianism in cases in which the agent has an opinion only about a finite set of possible worlds. We do not derive **Uniform Distribution** from one of the four precise versions of the Accuracy norm stated above, but from a stronger norm called Minimize, which we state below.

Minimize does not exactly employ the notion of expected local inaccuracy measure but something like an epistemic forerunner of it. This is one reason why we do not regard Minimize to be on equal terms with Accuracy (Synchronic expected local) and Accuracy (Diachronic expected local), which are used to derive the core tenets of Bayesianism. The other reason is that **Uniform Distribution** follows from Minimize a bit *too* easily, which is in contrast with the other proofs we present. So we do not insist on **Uniform Distribution**, since we do not see how we could—e.g., if you want to use a non-uniform prior belief function, maybe in order to make sure you can learn inductively, then so be it.[9] In any case, the normative claim in question is as follows:

> Minimize Suppose I is a legitimate local inaccuracy measure and suppose that E is the set of worlds that are epistemically possible for the agent. Then the agent ought to have a belief function b such that, for all $A \subseteq W$ and every $x \in \mathbb{R}_0^+$,
>
> $$\sum_{w \in E} I(A, w, b(A)) \leq \sum_{w \in E} I(A, w, x)$$

The sum in Minimize might appear to be given by an expected inaccuracy measure for a uniform belief function b, such that $b(w) = 1$ for all $w \in W$. Thus, it might seem that **Uniform Distribution** is *presupposed* by Minimize, rather than implied by it, as we claim. But this would be the wrong interpretation: instead, Minimize should be taken to express the epistemic goal of being as accurate as possible in a situation where the agent does not have a belief function at her disposal that she can use to assess her own expected inaccuracy; thus, *a fortiori*, she does not have a uniform belief function by which to do this.

Note that the nature of the belief function b that minimizes $\sum_{w \in W} I(A, w, b(A))$ depends on the local inaccuracy measure I; and the choice and justification of this in turn partially depends on the geometric framework we have determined in the prequel. So the prior belief function that a rational agent is bound to choose will reflect formal properties of the geometrical representation that we simply took for granted in the last section. Fair enough—that's how it goes with presuppositions.

Granted Minimize, **Uniform Distribution** follows by the following theorem:

Theorem 4 *Suppose b is a belief function, $E \subseteq W$, and I a quadratic local inaccuracy measure. Then the following two propositions are equivalent:*

(i) *For all $A \subseteq W$ and all $x \in \mathbb{R}_0^+$,*

$$\sum_{w \in E} I(A, w, b(A)) \leq \sum_{w \in E} I(A, w, x)$$

(ii) *For all $A \subseteq W$,*

$$b(A) = \frac{|A \cap E|}{|E|}$$

Again, the proof is given in the appendix (section 9.1.3).

[9] We thank Dorothy Edgington for this Carnapian point.

6.4 Hence, Bayesianism

This concludes our justification of the main tenets of Bayesianism in the case of agents who hold opinions concerning only a finite set of possible worlds. On the assumption of Local Inaccuracy Measures, we derived the normative claims of **Probabilism** and **Conditionalization** from the normative claims of the synchronic and diachronic local versions of the Accuracy norm, respectively. Moreover, if Minimize is accepted as well, then **Uniform Distribution** follows, too.

In the prequel, we derived Local Inaccuracy Measures by three separate arguments, each of which turned on excluding a certain sort of dilemma. As we promised in section 3, our argument for all of the Bayesian tenets turns ultimately on the epistemic virtue of a single goal, namely, the goal of having accurate belief functions, i.e., the Accuracy principle, in conjunction with the (internalistically) valid Ought-Can principle and the geometric framework that we presupposed in the present paper and which we explained in the prequel.

7 Jeffrey's updating rule

Before we turn to the general prospects of this theory and to the proofs of our central theorems, we investigate the status of **Jeffrey Conditionalization** in the context of the Accuracy norm. We show that it sometimes violates one of the instances of this norm, and we describe the updating rule that satisfies that instance.

As we noted above, Jeffrey's aim was to give an updating rule that covers those scenarios in which an agent obtains evidence corresponding to a format of side constraints other than those considered by **Conditionalization**.[10] In the cases covered by **Conditionalization**, the agent obtains evidence between times t and t' that restricts the set of worlds that are epistemically possible for her. On the other hand, in the cases covered by **Jeffrey Conditionalization**, her evidence does not rule out any possible worlds, but it does impose constraints on the belief function that the agent adopts at t'. These constraints are given in the following form: Suppose $\{E_1, ..., E_m\}$ is a partition of W and suppose that $q_1, ..., q_m \in \mathbb{R}_0^+$ are such that $q_1 + \ldots + q_m = 1$; then, for each $i = 1, ..., m$, $b_{t'}(E_i) = q_i$.[11]

However, as we will see below, Jeffrey's rule violates the version of the Accuracy norm that governs updating in the situations he considers. What is this norm? One might think at first that it is Accuracy (Diachronic expected local), the norm from which **Conditionalization** was derived above (section 6.2). After all, this norm governs exactly the sort of situation that interested Jeffrey. However, this norm cannot be satisfied in all the situations in which **Jeffrey Conditionalization** applies.[12] Thus, we retreat to the strictly weaker norm Accuracy (Diachronic expected global). This demands the following: when an agent's evidence imposes the constraints described above, the agent's belief function $b_{t'}$ at t' must satisfy those constraints and it must be minimal amongst the belief functions that satisfy those constraints with respect to its expected *global* inaccuracy by the lights of b_t, relative to a quadratic global inaccuracy measure G, and over the set of possible worlds that are epistemically possible at t.

[10] See p. 3 for the formal details.

[11] This is not the most general form of constraints of this sort. More generally, the E_is may not be pairwise disjoint, in which case the value of $q_1 + ... + q_m$ need not be 1. However, Jeffrey did not consider this case, and we postpone its consideration for another time.

[12] We prove this fact in section 9.3 of the appendix.

To introduce the norm that follows from Accuracy (Diachronic expected global) in the Jeffrey cases, let us consider two very natural ways in which one might try to satisfy the constraints imposed by the evidence in those cases:[13]

(i) On the first, we specify, for each member E_i of the partition, a constant c_i. And we obtain the degree of credence in each world $w \in E_i$ at t' by taking the degree of credence in w at t and *multiplying* it by c_i. That is, for $w \in E_i$,

$$b_{t'}(\{w\}) = c_i \cdot b_t(\{w\})$$

It is straightforward to see that, if $b_{t'}$ is to satisfy the constraints, there is only one way to define the constant c_i, namely, $c_i = \frac{q_i}{b_t(E_i)}$. Doing this gives **Jeffrey Conditionalization**.

(ii) On the second, we specify, for each E_i, a constant d_i. And we obtain the degree of credence in each world $w \in E_i$ at t' by taking the degree of credence in w at t and *adding* d_i to it. That is, for $w \in E_i$,

$$b_{t'}(\{w\}) = b_t(\{w\}) + d_i$$

It is straightforward to see that, if $b_{t'}$ is to satisfy the constraints, there is only one way to define the constant d_i, namely, $d_i = \frac{q_i - b_t(E_i)}{|E_i|}$. However, there is no guarantee that, on this definition, $b_t(\{w\}) + d_i$ is non-negative. Indeed, in some cases, it will be negative. We avoid this consequence as follows: in such cases, we let $b_{t'}(\{w\}) = 0$, for some worlds in E_i; and we seek a different value of d_i so that, for the remaining worlds in E_i, $b_{t'}(\{w\}) = b_t(\{w\}) + d_i$. Thus, we want our new value for d_i to be such that

(a) If $b_t(\{w\}) + d_i > 0$, then $b_{t'}(\{w\}) = b_t(\{w\}) + d_i$

(b) If $b_t(\{w\}) + d_i \leq 0$, then $b_{t'}(\{w\}) = 0$

(c) $\sum_{w \in E_i} b_{t'}(\{w\}) = q_i$.

It is straightforward to show that there is such a constant d_i and that this constant is unique. Defining d_i to be this constant, we obtain an alternative to **Jeffrey Conditionalization**:

$$b_{t'}(\{w\}) = \begin{cases} b_t(\{w\}) + d_i & \text{if } b_t(\{w\}) + d_i > 0 \\ 0 & \text{if } b_t(\{w\}) + d_i \leq 0 \end{cases}$$

We state this as a norm below and justify it by proving that it is the updating rule to which Accuracy (Diachronic expected global) gives rise in Jeffrey cases.

Here is the norm:

> **Alternative Jeffrey Conditionalization** Suppose that, between t and t', an agent obtains evidence that leads her to impose the following constraints on her belief function $b_{t'}$ at t': for each $i = 1, ..., m$, $b_{t'}(E_i) = q_i$.
>
> Then, for each $i = 1, ..., m$, define d_i as above: that is, let d_i be the unique real number such that
>
> $$\sum_{\{w \in E_i : b(\{w\}) + d_i > 0\}} b(\{w\}) + d_i = q_i$$

[13]We greatly appreciate the help provided by Alan Hájek and Kenny Easwaran in making our formulation and presentation of our alternative updating rule as intuitive as possible.

Then the agent ought to have belief function $b_{t'}$ at t' such that, for $w \in E_i$

$$b_{t'}(\{w\}) = \begin{cases} b_t(\{w\}) + d_i & \text{if } b_t(\{w\}) + d_i > 0 \\ 0 & \text{if } b_t(\{w\}) + d_i \leq 0 \end{cases}$$

And here is the justification:

Theorem 5 *Suppose G is a quadratic inaccuracy measure. If b is also a probability function, then we say that b* is feasible *if, for $i = 1, \ldots, m$, $b(E_i) = q_i$. Then the following two propositions are equivalent:*

(i) *$b_{t'}$ is feasible and, for any feasible probability function b,*

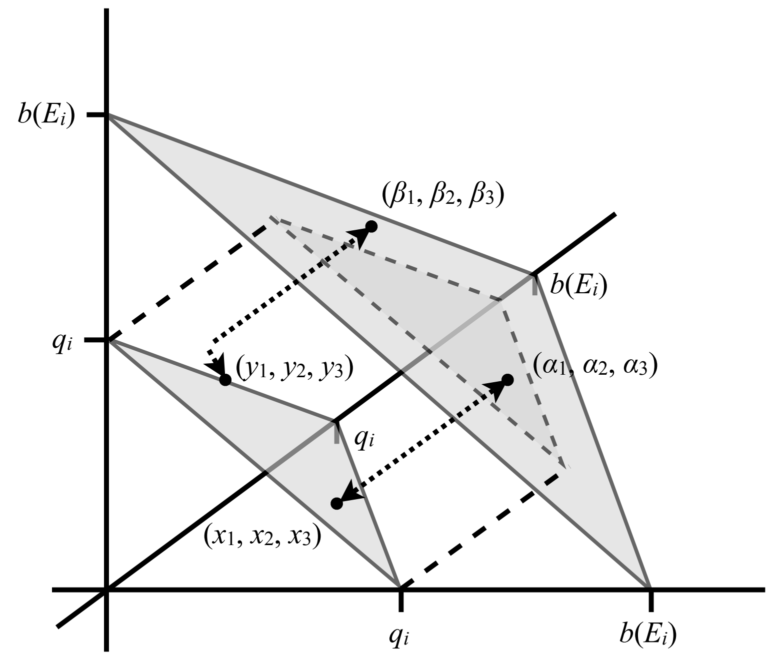
$$\mathrm{GExp}_{b_t}(G, W, b_{t'}) \leq \mathrm{GExp}_{b_t}(G, W, b)$$

(ii) *$b_{t'}$ is defined as in* **Alternative Jeffrey Conditionalization***.*

As in the other cases, we postpone the proof until the appendix (section 9).

Admittedly, the statement of this norm is not so transparent as Jeffrey's, but it follows from the proof of Theorem 5 that there is a natural geometric interpretation of the update rule that it describes. This is illustrated in Figure 1. Consider the element E_i of the partition. And suppose that with respect to E_i, the agent's belief function at t is represented by a point that lies within the larger grey triangle (as $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ do). Then our evidence imposes the constraint that her belief function at t' assigns q_i to E_i and hence must be represented by a point that lies within the smaller grey triangle. As it turns out, the point that minimizes global expected inaccuracy relative to this constraint is the point within the smaller grey triangle that lies closest to the point representing the original belief function, when that distance is measured by the Euclidean metric. Our statement of the updating rule above provides an analytic description of this point. Indeed, there are two cases: If the projection of the original belief function lies within the smaller grey triangle, then this projection already represents the belief function demanded by the updating rule (as is the case for $(\alpha_1, \alpha_2, \alpha_3)$ and its projection (x_1, x_2, x_3) in Figure 1). If it does not, the updated belief function is represented by the point on the smaller grey triangle that lies closest to that projection (as is the case for $(\beta_1, \beta_2, \beta_3)$ and (y_1, y_2, y_3) in Figure 1).

Having seen the updating rule sanctioned by the relevant version of Accuracy in Jeffrey cases, a number of its features deserve our attention. In section 7.1, we give an example to show that Jeffrey's rule results in belief functions with greater expected global inaccuracy than those given by our alternative rule. In section 7.2, we note that, as with Jeffrey's rule, the order in which compatible side constraints are imposed affects the posterior probability given by our rule: that is, our rule is non-commutative. We appeal to an insight of Marc Lange to show that this raises no objection. In section 7.3, we observe that **Conditionalization** is not a particular case of our rule, and we explain why this is as it should be. In section 7.4, we note that, unlike Jeffrey's rule, our rule can be used to raise probabilities from zero. And, in section 7.5, we reconsider the way in which the objective or 'quasi-logical' content of the diachronic versions of our Accuracy norm combine with the subjective or 'extra-logical' constraints C that are fed into it, and we thereby address a possible objection concerning the rigidity of conditional probabilities.



$b(E_i)$
$(\beta_1, \beta_2, \beta_3)$
$b(E_i)$
q_i
(y_1, y_2, y_3)
$(\alpha_1, \alpha_2, \alpha_3)$
q_i
(x_1, x_2, x_3)
q_i
$b(E_i)$

Figure 1:

7.1 The expected global inaccuracy of Jeffrey's rule

Suppose I see a person in the distance, and I know that it is one of three people: in w_1, it is Paul, who is male and has blond hair; in w_2, it is Jeff, who is male and has black hair; in w_3, it is Taj, who is female and has black hair. Suppose further that I know that the actual world is a member of $W = \{w_1, w_2, w_3\}$. At time t, I have the following belief function:

$$b_t(\{w_1\}) = \frac{1}{3}, \quad b_t(\{w_2\}) = \frac{1}{2}, \quad b_t(\{w_3\}) = \frac{1}{6}.$$

And between t and t', I have an experience that does not rule out any possible worlds, but which imposes the following side constraints on my beliefs:

$$b_{t'}(\text{the person is male}) = b_{t'}(\{w_1, w_2\}) = \frac{1}{2}.$$

Then Jeffrey's rule leads to the following values for $b_{t'}$:

$$b^J_{t'}(\{w_1\}) = \frac{1}{5}, \quad b^J_{t'}(\{w_2\}) = \frac{3}{10}, \quad b^J_{t'}(\{w_3\}) = \frac{1}{2}$$

If we let $G(\{w, b\}) = ||w - b_{\text{glo}}||^2$, then:

$$\begin{aligned}
&\mathrm{GExp}_{b_t}(G, W, b^J_{t'}) = \\
&\quad b_t(\{w_1\}) \left[\left(1 - \tfrac{1}{5}\right)^2 + \left(\tfrac{3}{10}\right)^2 + \left(\tfrac{1}{2}\right)^2\right] \quad + \\
&\qquad b_t(\{w_2\}) \left[\left(\tfrac{1}{5}\right)^2 + \left(1 - \tfrac{3}{10}\right)^2 + \left(\tfrac{1}{2}\right)^2\right] \quad + \\
&\qquad\quad b_t(\{w_3\}) \left[\left(\tfrac{1}{5}\right)^2 + \left(\tfrac{3}{10}\right)^2 + \left(1 - \tfrac{1}{2}\right)^2\right] = \tfrac{39}{50}
\end{aligned}$$

On the other hand, our rule leads to the following values for $b_{t'}$:[14]

$$b^A_{t'}(\{w_1\}) = \frac{1}{6}, \quad b^A_{t'}(\{w_2\}) = \frac{1}{3}, \quad b^A_{t'}(\{w_3\}) = \frac{1}{2}$$

So:

$$\begin{aligned}
&\mathrm{GExp}_{b_t}(\{G, W, b^A_{t'}\}) = \\
&\quad b_t(\{w_1\}) \left[\left(1 - \tfrac{1}{6}\right)^2 + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{2}\right)^2\right] \quad + \\
&\qquad b_t(\{w_2\}) \left[\left(\tfrac{1}{6}\right)^2 + \left(1 - \tfrac{1}{3}\right)^2 + \left(\tfrac{1}{2}\right)^2\right] \quad + \\
&\qquad\quad b_t(\{w_3\}) \left[\left(\tfrac{1}{6}\right)^2 + \left(\tfrac{1}{3}\right)^2 + \left(1 - \tfrac{1}{2}\right)^2\right] = \tfrac{35}{54}
\end{aligned}$$

Thus, the expected global inaccuracy of the feasible belief function that results from **Alternative Jeffrey Conditionalization** is lower than the expected global inaccuracy of the feasible belief function that results from **Jeffrey Conditionalization**.[15]

[14] To see this, notice that this is a case in which $b_{t'}(\{w\}) = b_t(\{w\}) + d_i$ with $d_i = \frac{q_i - b_t(E_i)}{|E_i|}$ does not result in negative values for $b_{t'}(\{w\})$; then calculate.

[15] Those readers aware of an important paper of Diaconis and Zabell [5] might be concerned that our result

7.2 Non-commutativity and the sameness of experience

We can extend the example of the previous section to show that, like **Jeffrey Conditionalization**, our rule is fundamentally non-commutative: that is, given a series of side constraints, and given successive applications of the rule that respect each of these side constraints in turn, the order in which the side constraints are imposed affects the final result; what's more, this remains true even when the side constraints are compatible in the sense that there are probability functions that satisfy them all at once.

One immediate—and, as we think, *valid*—reply to this is: so what? If we have to balance some of our pre-theoretical intuitions against a (hopefully) carefully crafted argument based on mathematical proof and established normative principles, it should be obvious which way to go. But let us examine the issue more closely and independently of such considerations.

In the previous section, we began with a belief function b_t and some side constraints on $b_{t'}$. Then we compared the effect of updating to a belief function that satisfies these side constraints using our rule and using Jeffrey's. We begin the following consideration by stating a new set of side constraints:

$$b_{t'}(\text{the person has black hair}) = b_{t'}(\{w_2, w_3\}) = \frac{3}{4}.$$

It is clear that this is compatible with the side constraints in the previous section, for there is at least one probability function that satisfies both.[16] However, as the calculations below show, if one begins with b_t, and then imposes the side constraint from the previous section, and then the side constraint from this section, our rule demands a belief function that differs from the belief function it demands if the order is reversed:[17]

(1) First, impose $b_{t'}(\{w_1, w_2\}) = \frac{1}{2}$:

$$b_{t'}(\{w_1\}) = \frac{1}{6}, \quad b_{t'}(\{w_2\}) = \frac{1}{3}, \quad b_{t'}(\{w_3\}) = \frac{1}{2}$$

Second, impose $b_{t''}(\{w_2, w_3\}) = \frac{3}{4}$:

$$b_{t''}(\{w_1\}) = \frac{1}{4}, \quad b_{t''}(\{w_2\}) = \frac{7}{24}, \quad b_{t''}(\{w_3\}) = \frac{11}{24}$$

(2) First, impose $b_{t'}(\{w_2, w_3\}) = \frac{3}{4}$:

$$b_{t'}(\{w_1\}) = \frac{1}{4}, \quad b_{t'}(\{w_2\}) = \frac{13}{24}, \quad b_{t'}(\{w_3\}) = \frac{5}{24}$$

---

is in tension with theirs. They prove that the updated belief function given by **Jeffrey Conditionalization** is the feasible belief function that is 'closest' to the original belief function on various plausible measures of closeness. However, there are differences between our approach and theirs. They seek the 'closest' function to the original function, whereas we seek the function whose expected inaccuracy is minimal by the lights of the original function. This said, it is a byproduct of our proof of Theorem 5 that the updated belief function given by our rule is also the feasible belief function that is closest to the original belief function on the *Euclidean* distance measure. But this is a measure of closeness that Diaconis and Zabell do not consider; if they had done so, they would have noticed that the updated belief function given by Jeffrey's updating rule is not the closest to the original belief function on this measure of closeness. We thank Brian Skyrms for pointing us to this literature.

[16] In fact, there is exactly one such probability function: $b(\{w_1\}) = \frac{1}{4}$, $b(\{w_2\}) = \frac{1}{4}$, $b(\{w_3\}) = \frac{1}{2}$.

[17] Again, this is a case in which $b_{t'}(\{w\}) = b_t(\{w\}) + d_i$ with $d_i = \frac{q_i - b_t(E_i)}{|E_i|}$ does not result in negative values for $b_{t'}(\{w\})$.

Second, impose $b_{t''}(\{w_1, w_2\}) = \frac{1}{2}$:

$$b_{t''}(\{w_1\}) = \frac{5}{48}, \quad b_{t''}(\{w_2\}) = \frac{19}{48}, \quad b_{t''}(\{w_3\}) = \frac{1}{2}$$

Some have taken the analogous result in the case of **Jeffrey Conditionalization** to be a flaw that is fatal for that rule ([26], [29], [6]). The objection, which is a *reductio*, is based on the following premise, which is made plausible in some toy story: in the situations described in (1) and (2) above, the first side constraint in (1) and the second side constraint in (2) have to be consequences of the same sensory experience; likewise, the second side constraint in (1) and the first side constraint in (2) have to be consequences of the same sensory experience. From this it follows that, on our rule or on Jeffrey's, one could obtain different belief functions simply by having the same sensory experiences, but in a different order. This, the objector claims, is counterintuitive and the *reductio* is complete.

The correct reply to this objection is already present in [7] and [26] (p. 197), but it is only stated explicitly by Marc Lange [20] (see also Wagner [30], pp. 274f for a similar point). It is simply that reversing the order of side constraints does not necessarily correspond to reversing sensory experiences. Being subject to the same side constraints due to what has been going on qualitatively in one's sensory organs is not sufficient for having the same sensory experiences; in order to individuate sensory experiences one also has to take into account the effect that the side constraints have on one's prior belief function and indeed the prior belief function on which they have that effect.

Thus, rather than being a flaw in our rule and in Jeffrey's, we should expect commutativity to fail for updating rules that apply to the cases Jeffrey considers. After all, on the view just explained, a particular side constraint corresponds to different sensory experiences if it is imposed on different prior belief functions. And we should not be surprised to find different sequences of sensory experience to give rise to different posterior belief functions.

7.3 Why **Conditionalization** is not a special case

Conditionalization is the special case of **Jeffrey Conditionalization** obtained by taking the partition $\{E_1 = E, E_2 = \neg E\}$ and letting $q_1 = 1$ and $q_2 = 0$. One might expect the same to hold of our rule, but this is not the case.

The reason is simple. There are two different sorts of constraint that new evidence can impose upon an agent's epistemic state: it can impose side constraints on the belief function that the agent should adopt in the light of the evidence; and it can restrict the set of worlds that are epistemically possible for the agent in the light of the evidence. **Jeffrey Conditionalization** is usually supposed to cover the former sort of situations; **Conditionalization** covers situations in which the latter sort of constraint is imposed. In the context of our theory, one deals in the former case with minimizing sums of the form

$$\sum_{w \in W} b(\{w\}) G(w, b')$$

where b is the (given) current belief function and where b' is unspecified except for the demand to satisfy the side constraints. In the latter case, however, one intends to minimize sums of the form

$$\sum_{w \in E} b(\{w\}) G(w, b')$$

in which b is again the (given) current belief function, b' is left completely unspecified, and where the sum is taken only over the worlds in E. If one tried to emulate conditionalization by the Jeffrey-type requirement that $b'(E) = 1$, then any permissible choice of b' would indeed assign 0 to $\neg E$, but this would still not necessarily be so for b; hence, in the emulation of conditionalization, inaccuracies with respect to worlds outside of E might still play a role, in contrast with the proper conditionalization case.

Thus, it is entirely appropriate that **Conditionalization** is not a special case of our rule. Learning a proposition with certainty is not the limiting case as the side constraints q_1 and q_2 on the partition $\{E_1 = E, E_2 = \neg E\}$ tend to 1 and 0, respectively, for as these values tend to zero, the set of epistemically possible worlds remains constantly W. Thus, the correct updating rule in the situations normally assumed to be covered by **Jeffrey Conditionalization** should not necessarily tend to **Conditionalization** in the limit.

7.4 Raising credences from zero

It is a well-known feature of **Jeffrey Conditionalization** that it cannot raise the probability of a proposition from zero. Thus, if this is the correct updating rule, we must forever assign zero to each proposition to which we currently assign zero. This, it has sometimes been argued, is too strong. It rules out the possibility of rationally coming to believe something that one once considered certainly false; yet this is surely possible.

It is a virtue of our rule that it does not have this consequence. Indeed, given a proposition A and a belief function b_t such that $b_t(A) = 0$, our rule applies even if the evidence an agent obtains results in the following side constraint on her belief function at t': $b_{t'}(A) = p > 0$ and $b_{t'}(\neg A) = 1 - p$. **Jeffrey Conditionalization** is not even defined in this case.

7.5 The logic vs. the art of judgement

In the light of the previous findings, let us reconsider one more time the norm on which the justification of our new rule of update is based[18]:

> Accuracy (Diachronic expected global) At time t', an agent ought to have a belief function that satisfies constraints C and is minimal amongst belief functions thus constrained with respect to expected global inaccuracy by the lights of your belief function at time t, relative to a legitimate global inaccuracy measure, and over the set of worlds that are epistemically possible for her at time t' given the constraints C.

The norm combines two kinds of constraints: (i) one ought to minimize one's expected global inaccuracy given certain parameters; (ii) the latter parameters are characterized in the way that they ought to satisfy C. What is the philosophical status of these constraints?

We regard (ii) as being given subjectively or 'extra-logically'. Within our theory, there is no room for justifying why C is such and such in a concrete application of Accuracy (Diachronic expected global) by a real-world agent. On the other hand, within the range of possibilities left open by C, it is a matter of epistemic rationality—a matter of getting as close to the truth as possible—to obey (i). In this sense, (i) is an objective or 'quasi-logical' constraint, but one that is conditional on the antecedently specified C condition. If C is e.g. such that one and

[18] This section benefited a lot from discussions with Carl Wagner, Richard Bradley, and Franz Dietrich.

only belief function $b_{t'}$ can satisfy it, then minimizing one's expected global inaccuracy *in the range of possibilities as determined by* C will be a trivial affair, and so be it according to our proposal.

As explained in the previous sections, conditionalization results from an application of Accuracy (Diachronic expected global) with an antecedent constraint C of the form 'restrict your set of epistemically possible worlds at t' to the set E'. In contrast, the new update rule that we have focused on this last part of our paper is due to an application of Accuracy (Diachronic expected global) with an antecedent constraint C of the form 'change your degrees of belief in a way such that for all i, E_i is believed at t' with degree q_i'. While these applications of Accuracy (Diachronic expected global) are clearly of broad interest, nothing prevents us from demanding other extra-logical constraints C to be satisfied at t', and consequently to search for rules of update which would minimize expected global inaccuracy in such circumstances.

For instance, one might be interested in a constraint C of the form 'change your degrees of belief in a way such that for all i, E_i is believed at t' with degree q_i, and furthermore Rigidity is satisfied, that is, $b_{t'}(A|E_i) = b_t(A|E_i)$ for all propositions $A \subseteq W$'. As noted before, Jeffrey's rule is the unique updating rule that leads to belief functions which satisfy this type of constraint. The fact that our rule of update differs from Jeffrey's should not be taken to imply that ours is 'logically valid' and Jeffrey's is not (or vice versa), but rather that the two rules are the objectively justified outcomes of solving one and the same epistemic problem—to get as close to the truth as possible—but *in two different problem spaces*.

This general line of reasoning could only be undermined by an argument which would show that some constraints C are 'more objective' or 'more logical' or 'more rational' than others. While we do not think that this can be ruled out completely, our theory does not offer any resources to put forward any plausible argument of that sort, and at least with respect to the question of whether to demand Rigidity or not, it is very hard to see that any such argument could be given at all. Indeed, we agree with Bradley [3] that sometimes Rigidity ought *not* to be demanded, in particular, when changes in belief give inferential grounds for changes in conditional belief. In principle, very much the same applies to the constraint that leads to simple conditionalization, however with one difference: in our theory, conditionalization is the objective consequence of the extra-logical constraint 'restrict your set of epistemically possible worlds at t' to the set E' in which Rigidity with respect to the partition $\{E, \neg E\}$ is not contained. It is only once the minimization problem is solved that Rigidity is seen to hold for the resulting solution strategy, that is, conditionalization. In this sense, the rigidity of plain conditionalization is 'more objective' than the rigidity of Jeffrey conditionalization. But of course even standard conditionalization might have to go if some other extra-logical constraint C is chosen, for whatever reason.

8 Some open questions

Obviously, our defence of Bayesianism in terms of minimizing expected inaccuracy leaves a lot of problems untouched. It is only fair to summarize the main open questions in the final section of this paper, posed as a challenge to future expansions of the theory:

- We asked this in the final section of this paper's prequel, but it is relevant again: How can the approach be extended to the case of an *infinite* set of worlds; in particular, to the case of non-denumerably many possible worlds? What role does countable additivity play in such extensions?

- Is it possible to develop a similar theory for *primitive conditional* belief functions, such as Popper measures, which allow for conditionalization on zero sets? Alternatively: What does a corresponding approach to *non-standard* probability measures look like?
- Is it possible to adapt this style of argument—by changing one of our presuppositions in some way—in order to justify other accounts of belief and belief update as well (such as, e.g., the Dempster-Shafer approach)?
- Given a different sort of constraint imposed by a piece of evidence, which updating rule does Accuracy (Diachronic expected global) prescribe? For instance:
  - Suppose an agent's evidence leads her to impose the following side constraints on $b_{t'}$: $b_{t'}(A) = p$ and $b_{t'}(B) = q$, where $A \cap B \neq \varnothing$. What is the prescribed rule of update?
  - Or suppose that $\{E_1, E_2, E_3\}$ is a partition on W, and the agent's evidence leads her to impose the following side constraints on $b_{t'}$: $b_{t'}(E_1) = kb_{t'}(E_2)$, where $k \in \mathbb{R}_0^+$. What is the prescribed rule of update in this case? (This is closely related to van Fraassen's well-known Judy Benjamin problem [27].)[19]
- It is easy to show that Accuracy (Diachronic expected local) is not always satisfiable given constraints C on the future belief function as used in **Jeffrey Conditionalization** (see section 9.3). Which belief functions at time t and which choices of epistemically possible worlds yield satisfiable instances of Accuracy (Diachronic expected local) for such C? In cases in which Accuracy (Diachronic expected local) cannot be satisfied, what do the belief functions look like which approximate Accuracy (Diachronic expected local) in the 'best possible' way, and how do these belief functions formally relate to the updating rule that we derived from Accuracy (Diachronic expected global)?

Answering these questions satisfactorily should lead not only to interesting extensions of our theory, but it should also help minimizing the inaccuracies of the theory as it stands.

9 Appendix: Proofs of Theorems 2, 3, 4, 5, and Accuracy (Diachronic expected local) again

9.1 Proofs of Theorems 2, 3, and 4

The proof of each of our theorems depends on the following lemma.

Lemma 6 *Suppose $I(A, w, x) = \lambda(\chi_A(w) - x)^2$. Suppose W is finite, b and b' are belief functions, $A, E \subseteq W$, and $\sum_{w \in E} b(w) \neq 0$. Then the following two propositions are equivalent:*

(i) *For all $A \subseteq W$ and all $x \in \mathbb{R}_0^+$,*

$$\sum_{w \in E} b(\{w\}) I(A, w, b'(A)) \leq \sum_{w \in E} b(\{w\}) I(A, w, x)$$

[19] We thank Alan Hájek and Kenny Easwaran for rightly urging us to include this into our list of open problems.

(ii) *For all* $A \subseteq W$*,*

$$b'(A) = \frac{\sum_{w \in A \cap E} b(\{w\})}{\sum_{w \in E} b(\{w\})}$$

Proof. By definition,

$$\sum_{w \in E} b(\{w\}) I(A, w, x) = \sum_{w \in E} b(\{w\}) \lambda (\chi_A(w) - x)^2$$

So,

$$\frac{d}{dx} \sum_{w \in E} b(\{w\}) I(A, w, x) = 2\lambda \left(x \sum_{w \in E} b(\{w\}) - \sum_{w \in E} b(\{w\}) \chi_A(w) \right)$$

Therefore,

$$\frac{d}{dx} \sum_{w \in E} b(\{w\}) I(A, w, x) = 0$$

if, and only if,

$$x = \frac{\sum_{w \in E} b(\{w\}) \chi_A(w)}{\sum_{w \in E} b(\{w\})} = \frac{\sum_{w \in A \cap E} b(\{w\})}{\sum_{w \in E} b(\{w\})}$$

Since $\sum_{w \in E} b(\{w\}) I(A, w, x)$ is a positive quadratic in the variable x, this extremum is a minimum, as required. $\square$

9.1.1 Proof of Theorem 2

Suppose b is a belief function and $E \subseteq W$, with $\sum_{w \in E} b(\{w\}) \neq 0$. Then, by Lemma 6, it suffices to show that

$$b(A) = \frac{\sum_{w \in A \cap E} b(\{w\})}{\sum_{w \in E} b(\{w\})},$$

if, and only if, b is a probability function on the power set of W and $b(\{w\}) = 0$ for $w \notin E$.

First, we prove the 'if' direction. We begin by showing that, if b is a probability measure and $b(\{w\}) = 0$ for $w \notin E$, then for all $A \subseteq W$,

$$b(A) = \frac{\sum_{w \in A \cap E} b(\{w\})}{\sum_{w \in E} b(\{w\})}.$$

If b is a probability measure and $b(\{w\}) = 0$ for $w \notin E$, then

$$1 = b(W) = \sum_{w \in W} b(\{w\}) = \sum_{w \in E} b(\{w\}) + \sum_{w \notin E} b(\{w\}) = \sum_{w \in E} b(\{w\})$$

So,

$$b(A) = \sum_{w \in A} b(\{w\}) = \sum_{w \in A \cap E} b(\{w\}) = \frac{\sum_{w \in A \cap E} b(\{w\})}{\sum_{w \in E} b(\{w\})}$$

as required.

Second, the 'only if' direction. That is, we show that, if b is a belief function and, for all $A \subseteq W$,

$$b(A) = \frac{\sum_{w \in A \cap E} b(\{w\})}{\sum_{w \in E} b(\{w\})},$$

then it follows that b satisfies (1), (2), and (3) below, the Kolmogorov axioms:

(1) *If* $A \subseteq W$*, then* $b(A) \geq 0$*.*

This is obvious, since $b : \mathcal{P}(W) \to \mathbb{R}_0^+$.

(2) $b(\varnothing) = 0$ *and* $b(W) = 1$*.*

$$b(\varnothing) = \frac{\sum_{w \in \varnothing \cap E} b(\{w\})}{\sum_{w \in E} b(\{w\})} = 0$$

and

$$b(W) = \frac{\sum_{w \in W \cap E} b(\{w\})}{\sum_{w \in E} b(\{w\})} = \frac{\sum_{w \in E} b(\{w\})}{\sum_{w \in E} b(\{w\})} = 1$$

(3) *If* $A, B \subseteq W$ *are disjoint, then* $b(A \cup B) = b(A) + b(B)$*.*

If $A, B \subseteq W$, then

$$\begin{aligned} b(A \cup B) &= \frac{\sum_{w \in (A \cup B) \cap E} b(\{w\})}{\sum_{w \in E} b(\{w\})} \\ &= \frac{\sum_{w \in A \cap E} b(\{w\})}{\sum_{w \in E} b(\{w\})} + \frac{\sum_{w \in B \cap E} b(\{w\})}{\sum_{w \in E} b(\{w\})} \\ &= b(A) + b(B) \end{aligned}$$

since $(A \cup B) \cap E = (A \cap E) \cup (B \cap E)$.

as required. Furthermore, if $w \notin E$, then obviously $b(\{w\}) = 0$. □

9.1.2 Proof of Theorem 3

Suppose b_t is a probability function, $I(A, w, x) = \lambda(\chi_A(w) - x)^2$, and $E \subseteq W$ with $b_t(E) \neq 0$. Then it follows immediately from Lemma 6 that, for all $A \subseteq W$,

$$\sum_{w \in E} b(\{w\}) I(A, w, x)$$

is minimal if, and only if,

$$x = \frac{b(A \cap E)}{b(E)} = b(A|E)$$

as required. □

9.1.3 Proof of Theorem 4

Suppose $I(A, w, x) = \lambda(\chi_A(w) - x)^2$. Then, in Lemma 6, let $b(\{w\}) = 1$ for all $w \in W$. Then

$$\sum_{w \in E} I(A, w, x)$$

is minimal if, and only if,

$$x = \frac{\sum_{w \in A \cap E} 1}{\sum_{w \in E} 1} = \frac{|A \cap E|}{|E|}$$

as required. □

9.2 Proof of Theorem 5

Suppose $\{E_1, \ldots, E_m\}$ is a partition of W. Suppose $0 \leq q_1, \ldots, q_m$ and $q_1 + \cdots + q_m = 1$. Suppose $G(w, b) = ||w - b_{\text{glo}}||^2$ and suppose that b_t is a probability function: let $\alpha_j = b_t(\{w_j\})$ for $j = 1, ..., n$. We wish to find the probability function $b_{t'}$ represented by the vector $(x_1^*, \ldots, x_n^*)$ such that the function

$$\text{GExp}_{b_t}(G, W, b_{t'}) = \sum_{w \in W} b_t(\{w\}) \cdot ||w - b_{t'}||^2 = \sum_{j=1}^{n} \alpha_j \left[x_1^2 + \cdots + x_{j-1}^2 + (x_j - 1)^2 + x_{j+1}^2 + \cdots + x_n^2\right]$$

is minimal at $(x_1^*, ..., x_n^*)$ relative to the following side constraints:

$$\begin{aligned} x_j &\geq 0 \text{ for } j = 1, \ldots, n \\ b_{t'}(E_i) &= q_i \text{ for } i = 1, \ldots, m \end{aligned}$$

We say that a vector $(x_1, ..., x_n)$ is *feasible* if it satisfies these constraints.

Now, we begin by reformulating the function we wish to minimize as follows:

$$\begin{aligned} \text{GExp}_{b_t}(G, W, b_{t'}) &= \sum_{j=1}^{n} \alpha_j \left(x_1^2 + ... + x_{j-1}^2 + (x_j - 1)^2 + x_{j+1}^2 + ... + x_n^2\right) \\ &= \sum_{j=1}^{n} \left(x_j^2(\alpha_1 + ... + \alpha_n) - 2\alpha_j x_j + \alpha_j\right) \\ &= \sum_{j=1}^{n} \left(x_j^2 - 2\alpha_j x_j + \alpha_j\right) \quad \text{since } \alpha_1 + ... + \alpha_n = 1 \\ &= \sum_{j=1}^{n} \left((x_j - \alpha_j)^2 - (\alpha_j^2 - \alpha_j)\right) \\ &= \sum_{j=1}^{n} (x_j - \alpha_j)^2 - \sum_{j=1}^{n} (\alpha_j^2 - \alpha_j) \end{aligned}$$

Now, it is clear that

$$\text{GExp}_{b_t}(G, W, (x_1^*, ..., x_n^*)) = \sum_{j=1}^{n} (x_j^* - \alpha_j)^2 - \sum_{j=1}^{n} (\alpha_j^2 - \alpha_j)$$

is minimal amongst the feasible vectors iff

$$\sum_{j=1}^{n} (x_j^* - \alpha_j)^2$$

is minimal amongst the feasible vectors. Thus, $b_{t'}$ is minimal just in case it is represented by the closest feasible vector $(x_1^*, ..., x_n^*)$ to $(\alpha_1, ..., \alpha_n)$ as measured by the Euclidean metric. But how do we find this closest feasible vector?

It is clear that

$$f((x_1^*, ..., x_n^*)) = \sum_{j=1}^{n}(x_j^* - \alpha_j)^2$$

is minimal amongst the feasible vectors iff, for each $i = 1, ..., m$, if $E_i = \{w_{l_1}, ..., w_{l_k}\}$, then

$$f_i((x_{l_1}^*, ..., x_{l_k}^*)) = \sum_{j=1}^{k}(x_{l_j}^* - \alpha_{l_j})^2$$

is minimal amongst those vectors $(x_{l_1}, ..., x_{l_k})$ for which $x_{l_1} + ... + x_{l_k} = q_i$ and $x_{l_j} \geq 0$ for all $j = 1, ..., k$. Thus, it suffices to solve the minimization problem separately for each element E_i of the partition.

We now give two different ways of showing that the vector given by **Alternative Jeffrey Conditionalization** solves each of these separate minimization problems. The first is our original proof and proceeds via the theory of convex quadratic programming and the Karush-Kuhn-Tucker (KKT) conditions that are central to that theory. The second is a purely geometric argument that we owe to Kenny Easwaran. We include both here since they exhibit quite different virtues. On the one hand, Easwaran's argument is simpler and requires less mathematical apparatus, but it is not clear how to generalize his approach so that it applies in updating situations that arise when different sorts of constraints are imposed on $b_{t'}$. On the other hand, our original argument from KKT conditions requires more powerful machinery, but it has the advantage of being fully general.

In what follows, we assume, without loss of generality, that $E_i = \{w_1, ..., w_k\}$. This will avoid unnecessarily complicated subscripts.

First, the argument from KKT conditions. The mathematical theorem we require is the following:[20]

Theorem 7 (KKT conditions) *Suppose $f, g_1, ..., g_m, h_1, ..., h_n : \mathbb{R}^k \to \mathbb{R}$ are smooth functions. Consider the following minimization problem:*

Minimize

$$f(x_1, ..., x_k)$$

relative to the following constraints:

$$\begin{aligned} g_i(x_1, ..., x_k) \leq 0 \quad & \textit{for } i = 1, ..., m \\ h_j(x_1, ..., x_k) = 0 \quad & \textit{for } j = 1, ..., n \end{aligned}$$

If $\vec{x^*} = (x_1^*, ..., x_k^*)$ is a (nonsingular) solution to this minimization problem, then there exist $\mu_1, ..., \mu_m$, $\lambda_1, ..., \lambda_n \in \mathbb{R}$ such that

$$\begin{aligned} \nabla f(\vec{x^*}) + \sum_{i=1}^{m}\mu_i \nabla g_i(\vec{x^*}) + \sum_{j=1}^{n}\lambda_j \nabla h_j(\vec{x^*}) &= 0 \\ \mu_i g_i(\vec{x^*}) &= 0 \quad \textit{for } i = 1, ..., m \\ \mu_i &\geq 0 \quad \textit{for } i = 1, ..., m \\ g_i(\vec{x^*}) &\leq 0 \quad \textit{for } i = 1, ..., m \\ h_i(\vec{x^*}) &= 0 \end{aligned}$$

[20] For a proof of this theorem together with a discussion of its uses, see §3.3 and §3.4 of [22].

If, furthermore, f and g are convex functions, then the existence of $\mu_1, ..., \mu_m$, $\lambda_1, ..., \lambda_n \in \mathbb{R}$ is sufficient for a solution to the minimization problem. If f is strictly convex, then their existence is sufficient for a unique solution.

Stated in the form used in the theorem, here is the problem we must solve:

Minimize

$$f_i(x_1, ..., x_k) = \sum_{j=1}^{k} (x_j - \alpha_j)^2$$

relative to the following constraints:

$$\begin{array}{rcl} g_j(x_1, ..., x_k) = -x_j & \leq & 0 \quad \text{for } j = 1, ..., k \\ h(x_1, ..., x_k) = x_1 + ... + x_k - q_i & = & 0 \end{array}$$

Thus, since f_i, g_1, ..., g_k, h are smooth functions and since f_i is strictly convex, it is sufficient for $(x_1^*, ..., x_k^*)$ to be a unique solution to this minimization problem that $(x_1^*, ..., x_k^*)$ satisfies the constraints and there exist $\mu_1, ..., \mu_k, \lambda \in \mathbb{R}$ such that, for all $j = 1, ..., k$

(i) $\mu_j \geq 0$

(ii) $\mu_j x_j^* = 0$

(iii) $2x_j^* - 2\alpha_j - \mu_j + \lambda = 0$

Now, define d_i as in **Alternative Jeffrey Conditionalization** and let

$$x_j^* = \begin{cases} \alpha_j + d_i & \text{if } \alpha_j + d_i > 0 \\ 0 & \text{if } \alpha_j + d_i \leq 0 \end{cases}$$

In order to prove Theorem 5, it suffices to show that $(x_1^*, ..., x_k^*)$ thus defined satisfies the constraints, and that there are $\mu_1, ..., \mu_k, \lambda \in \mathbb{R}$ that satisfy (i)–(iii). It is straightforward to see that $(x_1^*, ..., x_k^*)$ satisfies the constraints. Now define

$$\lambda = -2d_i$$

and

$$\mu_j = \begin{cases} 0 & \text{if } \alpha_j + d_i > 0 \\ -2(\alpha_j + d_i) & \text{if } \alpha_j + d_i \leq 0 \end{cases}$$

It is straightforward to see that (i)–(iii) then hold. This completes our first proof of Theorem 5.

We turn now to Kenny Easwaran's geometric proof. First, we note that, since the set of feasible vectors is closed and bounded, and since the Euclidean distance from $(\alpha_1, ..., \alpha_k)$ to $(x_1, ..., x_k)$ is a continuous function of $(x_1, ..., x_k)$, there is at least one feasible vector $(x_1^*, ..., x_k^*)$ such that the Euclidean distance from $(\alpha_1, ..., \alpha_k)$ to that vector is minimal. Next, we use the following lemma to identify the unique such feasible vector.

Lemma 8 *Suppose $(x_1, ..., x_k)$ is feasible. Then, if $x_b > 0$ and $x_a - \alpha_a < x_b - \alpha_b$, then the distance from $(\alpha_1, ..., \alpha_k)$ to $(x_1, ..., x_k)$ is not minimal amongst the feasible vectors.*

Proof. Suppose $(x_1, ..., x_a, ..., x_b, ..., x_k)$ is feasible and suppose that $x_b > 0$ and $x_a - \alpha_a < x_b - \alpha_b$. Then let ε be a positive real number such that

$$\varepsilon < x_b \quad \text{and} \quad \varepsilon < (x_b - \alpha_b) - (x_a - \alpha_a)$$

Then $(x_1, ..., x_a + \varepsilon, ..., x_b - \varepsilon, ..., x_k)$ is also feasible. Moreover, a quick calculation shows that it is closer to $(\alpha_1, ..., \alpha_k)$ than is $(x_1, ..., x_a, ..., x_b, ..., x_k)$. This completes the proof of the lemma. □

With this in hand, we can identify the unique vector whose distance from $(\alpha_1, ..., \alpha_k)$ is minimal. We require two corollaries to the lemma. **First corollary**: Suppose $(x_1^*, ..., x_k^*)$ is minimal; then there is a real number d_i such that, if $x_a^* > 0$, then $x_a^* = \alpha_a + d_i$. *Proof*: Suppose $x_a^*, x_b^* > 0$. Then, by the lemma, it must be that $x_a - \alpha_a = x_b - \alpha_b$. Thus, there is $d_i = x_a - \alpha_a = x_b - \alpha_b$, as required. **Second corollary**: Suppose $(x_1^*, ..., x_k^*)$ is minimal; and suppose that, whenever $x_a^* > 0$, we have $x_a^* = \alpha_a + d_i$; then, whenever $\alpha_a + d_i > 0$, we have $x_a^* = \alpha_a + d_i$. *Proof*: Suppose not. That is, suppose $\alpha_a + d_i > 0$ and $x_a^* \neq \alpha_a + d_i$. Then, by previous corollary, $x_a^* = 0$. Now suppose $x_b > 0$. Then

$$x_a^* - \alpha_a = -\alpha_a = -(\alpha_a + d_i) + d_i < d_i = x_b^* - \alpha_b$$

Thus, by the lemma, $(x_1^*, ..., x_k^*)$ is not minimal. This contradicts the assumption, as required.

From these two corollaries to the lemma, we have that, if $(x_1^*, ..., x_k^*)$ is minimal, there is d_i such that, for all $j = 1, ..., k$

(a) If $\alpha_j + d_i > 0$, then $x_j^* = \alpha_j + d_i$

(b) If $\alpha_j + d_i \leq 0$, then $x_j^* = 0$

(c) $\sum_{j=1}^{k} x_j^* = q_i$.

That is, the vector to which **Alternative Jeffrey Conditionalization** gives rise is the closest vector to $(\alpha_1, ..., \alpha_k)$, as required. This completes the second proof of Theorem 5, due to Kenny Easwaran. □

9.3 Accuracy (Diachronic expected local) cannot always be satisfied in Jeffrey situations

In sections 9.1.1 and 9.1.2, we derived **Probabilism** and **Conditionalization** from the *local* synchronic and diachronic versions of Accuracy, respectively. But we derived our alternative to Jeffrey's rule from the *global* diachronic version of Accuracy (along with **Probabilism**, which is guaranteed by Accuracy (Synchronic expected local)). Above, we mentioned why this is: the *local* version cannot always be satisfied in the situations to which **Jeffrey Conditionalization** claims to apply. In this section, we prove this.

First, recall that Accuracy (Diachronic expected local) entails Accuracy (Diachronic expected global). That is, any belief function that satisfies the former satisfies the latter. Also, we know which belief function satisfies the latter, in virtue of Theorem 5, proved above. Thus, it will suffice to describe a Jeffrey situation in which the belief function that satisfies Accuracy (Diachronic expected global) does not satisfy Accuracy (Diachronic expected local).

Consider again the example of section 7.1. That is: $W = \{w_1, w_2, w_3\}$ and

$$b_t(\{w_1\}) = \frac{1}{3}, \quad b_t(\{w_2\}) = \frac{1}{2}, \quad b_t(\{w_3\}) = \frac{1}{6}.$$

We then impose the following constraints: $b_t(\{w_1, w_2\}) = \frac{1}{2}$. Then our updating rule gives:

$$b^A_{t'}(\{w_1\}) = \frac{1}{6}, \quad b^A_{t'}(\{w_2\}) = \frac{1}{3}, \quad b^A_{t'}(\{w_3\}) = \frac{1}{2}.$$

Let $I(A, w, x) = (\chi_A(w) - x)^2$, and consider the expected local inaccuracy of the degree of credence $b^A_{t'}(\{w_1\})$ in the singleton proposition $\{w_1\}$ by the lights of b_t, relative to I, and over all possible worlds in W:

$$\mathrm{LExp}_{b_t}(I, \{w_1\}, W, b^A_{t'}(\{w_1\})) = \frac{1}{3}\left(1 - \frac{1}{6}\right)^2 + \frac{1}{2}\left(-\frac{1}{6}\right)^2 + \frac{1}{6}\left(-\frac{1}{6}\right)^2 = \frac{2}{8}$$

Now consider the following belief function:

$$b^C_{t'}(\{w_1\}) = \frac{1}{3}, \quad b^C_{t'}(\{w_2\}) = \frac{1}{6}, \quad b^C_{t'}(\{w_3\}) = \frac{1}{2}$$

Then:

$$\mathrm{LExp}_{b_t}(I, \{w_1\}, W, b^C_{t'}(\{w_1\})) = \frac{1}{3}\left(1 - \frac{1}{3}\right)^2 + \frac{1}{2}\left(-\frac{1}{3}\right)^2 + \frac{1}{6}\left(-\frac{1}{3}\right)^2 = \frac{2}{9}$$

Thus,

$$\mathrm{LExp}_{b_t}(I, \{w_1\}, W, b^C_{t'}(\{w_1\})) < \mathrm{LExp}_{b_t}(I, \{w_1\}, W, b^A_{t'}(\{w_1\})).$$

As noted above, this suffices to show that Accuracy (Diachronic expected local) cannot be satisfied in all Jeffrey situations. □

Acknowledgements: We would like to thank F. Arntzenius, L. Bayón, R. Bradley, F. Dietrich, K. Easwaran, B. Fitelson (and his Berkeley reading group), D. Edgington, A. Hájek, L. Horsten, F. Huber, J. Joyce, W. Myrvold, S. Okasha, G. Schurz, T. Seidenfeld, B. Skyrms, C. Wagner, R. Williams, J. Williamson, B. van Fraassen for their comments on earlier versions of this paper. Hannes Leitgeb would like to thank the Leverhulme Trust and the Alexander von Humboldt Foundation for their generous support of this work. Richard Pettigrew would like to thank the British Academy with whom he was a postdoctoral fellow during work on this paper.

References

[1] J. O. Berger. *Statistical Decision Theory and Bayesian Analysis*. Springer, 1985.

[2] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, Cambridge, 2004.

[3] R. Bradley. Radical Probabilism and Bayesian Conditioning. *Philosophy of Science*, 72:342–364, 2005.

[4] B. de Finetti. Sul significato soggettivo della probabilita. *Fundamenta Mathematicae*, 17:298–329, 1931.

[5] P. Diaconis and S. L. Zabell. Updating Subjective Probability. *Journal of the American Statistical Association*, 77(380):822–830, 1982.

[6] F. Döring. Why Bayesian Psychology is Incomplete. *Philosophy of Science (Proceedings)*, 66:S379–S389, 1999.

[7] H. Field. A Note on Jeffrey Conditionalization. *Philosophy of Science*, 45:361–367, 1978.

[8] R. Foley. The Epistemology of Belief and the Epistemology of Degrees of Belief. *American Philosophical Quarterly*, 29:111–124, 1992.

[9] A. Gibbard. Rational Credence and the Value of Truth. In T. Gendler and J. Hawthorne, editors, *Oxford Studies in Epistemology*, volume 2, pages 143–164. Oxford University Press, Oxford, 2008.

[10] H. Greaves and D. Wallace. Justifying Conditionalization: Conditionalization Maximizes Expected Epistemic Utility. *Mind*, 115(459):607–632, 2006.

[11] A. Hájek. Arguments for—or Against—Probabilism?. *The British Journal for the Philosophy of Science*, 59(4): 793–819, 2008.

[12] G. Harman. *Change in View: Principles of Reasoning.* MIT Press, Cambridge, Mass., 1986.

[13] F. Huber and C. Schmidt-Petri, editors. *Degrees of Belief.* Synthese Library, volume 342, Springer, 2009.

[14] E. T. Jaynes. *Probability Theory: The Logic of Science.* Cambridge University Press, Cambridge, 2003.

[15] R. Jeffrey. *Logic of Decision.* McGraw-Hill, New York, 1965.

[16] H. Jeffreys. *Theory of Probability.* Oxford University Press, Oxford, 1998.

[17] J. M. Joyce. A Nonpragmatic Vindication of Probabilism. *Philosophy of Science*, 65(4):575–603, 1998.

[18] J. M. Joyce. Accuracy and Coherence: Prospects for an Alethic Epistemology of Partial Belief. In Huber and Schmidt-Petri [13], pages 263–297.

[19] M. Lange. Calibration and the Epistemological Role of Bayesian Conditionalization. *The Journal of Philosophy*, 96(6):294–324, 1999.

[20] M. Lange. Is Jeffrey Conditionalization Defective by Virtue of Being Non-Commutative? Remarks on the Sameness of Experience. *Synthese*, 123:393–403, 2000.

[21] D. K. Lewis. Why Conditionalize? In *Papers in Metaphysics and Epistemology.* Cambridge University Press, Cambridge, 1999.

[22] P. Pedregal. *Introduction to Optimization.* Springer-Verlag, 2003.

[23] K. R. Popper. *The Logic of Scientific Discovery, Revised Edition.* Hutchinson, London, 1968.

[24] F. P. Ramsey. Truth and Probability. *The Foundations of Mathematics and Other Logical Essays*, pages 156–198, 1931.

[25] G. Schurz and H. Leitgeb. Finitistic and Frequentistic Approximation of Probability Measures with or without σ-Additivity. *Studia Logica*, 89(2):257–283, 2008.

[26] B. Skyrms. *Choice and Chance.* Wadsworth Publishing Co., Belmont, CA, 3rd edition, 1986.

[27] B. C. van Fraassen. A Problem for Relative Information Minimizers. *The British Journal for the Philosophy of Science*, 32(4):375–379, 1981

[28] B. C. van Fraassen. A Demonstration of the Jeffrey Conditionalization Rule. *Erkenntnis*, 24(1):17–24, 1986.

[29] B. C. van Fraassen. *Laws and Symmetry.* Oxford University Press, Oxford, 1989.

[30] C. G. Wagner. Probability Kinematics and Commutativity. *Philosophy of Science*, 69:266–278, 2002.

[31] P. M. Williams. Bayesian Conditionalization and the Principle of Minimum Information. *The British Journal for the Philosophy of Science*, 31:131–144, 1980.

[32] J. Williamson. Motivating Objective Bayesianism: From Empirical Constraints to Objective Probabilities. In W. L. Harper and G. R. Wheeler, editors, *Probability and Inference: Essays in Honor of Henry E. Kyburg Jr.*, pages 155–183. College Publications, London, 2007.