Appendix 1: Logic

The only symbolic logic used in this book is a small part of *propositional logic*, also called *sentential logic* or *Boolean logic*. In this appendix, I review the relevant part of this simple area of logic and clarify some notation and terminology. This appendix is not an introduction to logic; various important fine points and distinctions are not be mentioned. But I hope this will suffice as an introduction to the basic ideas in the elementary part of logic used in this book, and as a clarification of the logical terminology and symbols used in this book.

The basic entities of the formal propositional calculus are usually called the *propositions* and the propositional *connectives* (and the language of propositional logic usually includes *punctuation marks*, usually parentheses, that are used to avoid ambiguity of grouping when "propositions" are "connected" in complex ways).

In this book, it is *factors* (or *properties*, or *types*) that play the role of the so-called *propositions* of propositional logic. The abstract and formal propositional calculus can be *interpreted* as applying to propositions in a number of ways in which the term "proposition" could be understood. For example, we could think of propositions as sentences (which may be understood as concrete linguistic entities such as utterances or inscriptions). Or we could think of them as statements (understood in such a way that many sentences can all be used to "make" the same statement, and the same sentence, if used in different contexts, would make different statements). Or we can think of propositions as *sets of "possible worlds"* (as many philosophers and logicians have done in studies of modalities, such as possibility and necessity, and various kinds of logical relations, such as counterfactual conditional connections). And the propositions can also be understood as factors (or properties, or types).

In interpreting the abstract and formal propositional calculus, we can think of propositions *either* as something like sentences or statements, in which case they are, roughly speaking, *true or false in a given situation, or* as factors (properties or types), in which case they are *exemplified or not exemplified in a given instance*. In this appendix, which is a general review of some *formal* aspects of *abstract* propositional logic, I will use the term "proposition" to refer to the entities that play this role in the calculus. And I will generally speak of propositions as being either true or false, rather than of factors as being either exemplified or not exemplified. Formally, the idea of the truth or falsity of a given sentence or statement in a situation (or a possible world) is quite parallel to the idea of the presence or absence of a given factor (or property or type) in an instance. For example, a factor is exemplified or not exemplified in a situation, according to whether the statement asserting it is exemplified is true or false in the situation. And the formal parallelism remains intact with the introduction of the propositional connectives.

The *connectives* of propositional logic, sometimes called *sentential connectives* or *truth-functional connectives* or *Boolean connectives*, can be used to form "new" propositions from "old" ones. The "new" propositions are called *truth-functional compounds*, or *Boolean compounds*, of the "old" propositions. The most usual connectives (and the only propositional connectives used in this book) are the ones for which I use the following three symbols (as is most standard): "$\sim$", "&", and "$\vee$". These represent negation ("not"), conjunction ("and"), and disjunction "or, or both").

Let X and Y be any propositions. Then, understood as sentences or statements, X and Y are entities that are either true or false, in any given situation. (Alternatively, X and Y

can be factors, which are entities that are, in any given instance, either exemplified or not exemplified.) Then: (1) $\sim X$, the negation of X, is the proposition that is true in a situation just in case (if and only if) X is false in the situation (or it is the factor that is exemplified in an instance just in case X is not exemplified in the instance); (2) $X\&Y$, the *conjunction* of X and Y, is the proposition that is true in a situation just in case both X is true in the situation and Y is true in the situation (or it is the factor that is exemplified in an instance just in case both X is exemplified in the instance and Y is exemplified in the instance); and (3) $X \vee Y$, the *disjunction of* X and Y, is the proposition that is true in a situation just in case at least one of X and Y is true in the situation (or it is the factor that is exemplified in an instance just in case at least one of X and Y is exemplified in the instance).

X is sometimes called the *negatum* of $\sim X$. X and Y are called the *conjuncts* of $X\&Y$. And X and Y are called the *disjuncts* of $X \vee Y$. More than two propositions can be *conjoined:* Their conjunction is true (exemplified) if all the conjuncts are true (exemplified). And more than two propositions can be *disjoined:* Their disjunction is true (exemplified) if at least one of the disjuncts is true (exemplified).

Three important kinds of propositions are the tautologies, the contradictions, and the contingent propositions. A proposition is a *tautology* (or *logically true*) if it cannot be false (it cannot not be exemplified). A proposition is a *contradiction* (or *logically false*) if it cannot be true (it cannot be exemplified). And a proposition is *contingent* (or *logically indeterminate*) if it is both possible for it to be true (exemplified) and possible for it to be false (be not exemplified). If X is any proposition, then examples of tautologies are $X \vee \sim X$ and $\sim(X\&\sim X)$, and examples of contradictions are $X\&\sim X$ and $\sim(X \vee \sim X)$. (Hereafter, I will not add the parenthetical "factor *exemplified* in an instance" after "proposition *true* in a situation", or "factor *not exemplified* in an instance" after "proposition *false* in a situation" – the parallel is always the same.)

Three important relations that can obtain between two propositions are those of implication, equivalence, and independence. A proposition X *implies* (or *logically implies*) a proposition Y if it is not possible for X to be true while Y is false (in the same situation). Propositions X and Y are *equivalent* (or *logically equivalent*) if they must be either both true or both false (they cannot differ from each other with respect to truth and falsity in a given situation). Another way of putting this is to say that X and Y are equivalent if each implies the other. Finally, two propositions are *independent* (*logically independent*) if all four combinations of the truth and falsity of X and the truth and falsity of Y are possible. Another way of putting this is to say that X and Y are independent if neither of X and $\sim X$ implies either of Y or $\sim Y$ (and vice versa, to be redundant).

A *set of propositions* is said to be *closed* under the propositional connectives ($\sim$, &, $\vee$) if it contains all Boolean compounds ($\sim X$, $X \& Y$, $X \vee Y$) of propositions (X and Y) that it contains. And the *closure* of a set of propositions is the smallest set that contains the given set and is closed under the propositional connectives.

Propositions in a set of propositions are *mutually exclusive* if at most one of them could be true: *The conjunction of any two or more of them is a contradiction (or logically false)*. Propositions in a set of propositions are *collectively exhaustive* if they cannot all be false: *The disjunction of all of them is a tautology (or logically true)*. A *partition* is a set of mutually exclusive *and* collectively exhaustive propositions. For any partition, exactly one proposition in it is true (in a given situation).

Relative to a set S of propositions that is closed under the usual connectives, a proposition X is *maximally specific* if X is a member of S and there is no proposition Y in S such that both (1) Y implies X and (2) Y is not equivalent to X – that is, any proposition Y in S that implies X is equivalent to X. If we assume an interpretation of propositions under which *logically equivalent* propositions are *identical* propositions (which is plausible for propositions understood as statements, as sets of possible worlds, or as factors, or properties or types, but *not* for propositions understood as sentences), then we could say that X is maximally specific relative to a closed set S if no proposition in S, other than X, implies X. *Relative to any set T of propositions, X is maximally specific* if X is maximally specific relative to the closure of T under the usual connectives.

Any element of a set of propositions is equivalent to a disjunction whose disjuncts are propositions that are maximally specific relative to the set. And a maximally specific proposition relative to a set is always equivalent to a conjunction whose conjuncts are all either members of the set or negations of members of the set. The set of propositions maximally specific relative to a set is always (*modulo equivalence* – that is, treating equivalent propositions as identical) a partition. If $S_1, \ldots, S_n$ are all partitions, then a proposition is maximally specific relative to the union of the S_i's if and only if it is equivalent to a conjunction $X_1 \& \ldots \& X_n$, where each X_i is a member of S_i.

Not every set of propositions is such that, relative to it, there exist maximally specific propositions; there are the "atomless Boolean algebras." A set of propositions that is closed under the usual connectives, together with the relation of implication, is one kind of Boolean algebra. A *Boolean algebra*, *B*, is simply any structure, $B = <S,\sim,\&,\vee,\Rightarrow,0,1>$, in which S plays the formally analogous role of a set of propositions that is closed under the operations formally analogous to the ways "$\sim$", "&", and "$\vee$" were described above, where $\Rightarrow$ corresponds to implication, and where for any element X of S, 0 is equivalent (and identical) to $X\&\sim X$ and 1 is equivalent (and identical) to $X\vee\sim X$.

A Boolean algebra is called *atomless* if, for every element X of it, there is another element Y of it that is "strictly less than" X. Understanding the algebra as a set of propositions, this means that Y implies X but X does not imply Y (so that $X \neq$ Y). Of course, all atomless Boolean algebras are infinite. And atomless Boolean algebras do not have any maximally specific elements. All finite Boolean algebras have maximally specific elements, called *atoms*, such that every member of the algebra is a disjunction of these atoms. A Boolean algebra is *complete* (or *sigma-additive*) if it contains all infinite conjunctions and disjunctions of its members, as well as the finite conjunctions and disjunctions of its members. All complete Boolean algebras have atoms such that every member of the algebra is a disjunction of atoms; not all infinite Boolean algebras are atomless.