# PUZZLE

You meet three individuals - one knight who always tells the truth, a knave who always lies, and a normal person who can do either. They know each other's identities.

A says "B is the normal one."

B says "No, C is the normal one."

C says "No, B is definitely the normal one."

Who is what?

# Formal Proofs and Boolean Logic

Wednesday, 15 September

# Constructing a Formal Proof

- In a proof you assume a set of premises, and work step by step to the desired conclusion (if the conclusion is a logical consequence of the premises!)
- Each step is justified by invoking a rule that is part of our formal system of deduction.
- In this class, we have been using Fitch but there are other systems of proof (deductive systems).

# Formal Proof Rules (Old)

- Reiteration

| | | |
|---|---|---|
| 1. P | | |
| 2. P | | Reit: 1 |

- *Ana Con (follows logically because of logical operators plus meaning of some TW predicates)

| | | |
|---|---|---|
| 1. Smaller(a,b) | | |
| 2. ¬Smaller(b,a) | | Ana Con 1 |

# FORMAL PROOF RULES (=)

- = Introduction

| | |
|---|---|
| 1. a=a | = Intro |

- = Elimination

| | |
|---|---|
| 1. Pred(a) | |
| 2. a=b | |
| 3. Pred(b) | = Elim: 1,2 |

# Formal Proof Rules (∧)

- ∧ Introduction

  From P and Q, we can infer P∧Q.

  | | |
  |---|---|
  | 1. P | |
  | 2. Q | |
  | 3. P ∧ Q | ∧ Intro: 1,2 |

- ∧ Elimination

  From P∧Q, we can infer P.

  | | |
  |---|---|
  | 1. P ∧ Q | |
  | 2. P | ∧ Elim: 1 |

# Formal Proof Rules (∨)

- ∨ Introduction

  From P, we can infer P∨Q.

  | | |
  |---|---|
  | 1. P | |
  | 2. P ∨ Q | ∨ Intro: 1 |

- ∨ Elimination

  Start with P∨Q. Assume P - get R. Assume Q - get R. Then you can infer R by ∨E.

# Proof by Cases

Example:

Cube(a) ∨ Dodec(a)

¬Tet(a)

1. Cube(a) ∨ Dodec(a)
2. Cube(a)
3. ¬Tet(a) Ana Con 2
4. Dodec(a)
5. ¬Tet(a) Ana Con 4
6. ¬Tet(a) ∨ Elim: 1,2-3,4-5

# ∧, ∨ Distribution Rules

- Distribution rules:

- $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$
- $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

# Nested Subproofs

- We can introduce any assumption you want any time in a proof by introducing a new scope line. (If you do so, make sure you know how to get rid of it).
- Some proofs require nested subproofs - subproofs inside other subproofs.
- Example - when you have two ∨Elims in the same proof.

# Formal Proof Rules (¬)

- ¬ Elimination
  From ¬¬P, we can infer P.

| 1. ¬¬P | |
| --- | --- |
| 2. P | ¬ Elim: 1 |

- **Incorrect** (not main connective)

| 1. ¬¬P ∨ Q | |
| --- | --- |
| 2. P ∨ Q | ¬ Elim: 1 |

# NESTED SUBPROOFS

$\neg\neg P \vee Q$

$R \vee \neg\neg S$

---

$(P \wedge R) \vee (P \wedge S) \vee (Q \wedge R) \vee (Q \wedge S)$

# RULES USING CONTRADICTIONS

- ¬ Introduction
  From showing P leads to ⊥, we can infer ¬P.

| | |
|---|---|
| \| 1. P | |
| \| … | |
| \| j. ⊥ | |
| k. ¬P | ¬ Intro: 1-j |

- Within a subproof we derive ⊥ from P; outside the subpro

# Rules Using Contradictions

- ⊥ Introduction
  From P and ¬P, we can infer ⊥.

| | |
|---|---|
| 1. P | |
| 2. ¬P | |
| 3. ⊥ | ⊥ Intro: 1, 2 |

# Reductio Ad Absurdum

Example:

$\neg(a=b \lor b=c)$

$a \neq b \land b \neq c$

| | | |
|---|---|---|
| 1. | $\neg(a=b \lor b=c)$ | |
| 2. | $a=b$ | |
| 3. | $a=b \lor b=c$ | $\lor$ Intro 2 |
| 4. | $\bot$ | $\bot$ Intro 1,3 |
| 5. | $a \neq b$ | $\neg$ Intro 2-4 |
| 6. | $b=c$ | |
| 7. | $a=b \lor b=c$ | $\lor$ Intro 6 |
| 8. | $\bot$ | $\bot$ Intro 1,7 |
| 9. | $b \neq c$ | $\neg$ Intro 6-8 |
| 10. | $a \neq b \land b \neq c$ | $\land$ Intro 5-9 |

# Rules Using Contradictions

- ⊥ Elimination

  From ⊥, we can infer absolutely whatever we want.

  | | | |
  |---|---|---|
  | 1. | ⊥ | |
  | 2. | BlueCheese(Moon) | ⊥ Elim: 1 |

- This is helpful when we want to eliminate a disjunct when we know that its negation is true.
- We don't technically need this rule; we could just use ¬ Intro and ¬ Elim.

# Rules Using Contradictions

Example: Disjunctive Syllogism

| | |
|---|---|
| $P \vee Q$ | |
| $\neg P$ | |
| $Q$ | |

| Step | Justification |
|---|---|
| 1. $P \vee Q$ | |
| 2. $\neg P$ | |
| 3. $P$ | |
| 4. $\bot$ | $\bot$ Intro 2,3 |
| 5. $Q$ | $\bot$ Elim 4 |
| 6. $Q$ | |
| 7. $Q$ | $\vee$ Elim 1,3-5,6-6 |