PROOF THAT
LOGIC IS AWESOME

Call the following sentence P: "If this conditional is true, then Logic is Awesome". I will now prove that P is in fact true. P is a conditional. I will assume its antecedent and then show that its consequent would follow. So we assume that "this conditional is true". "this conditional" refers to sentence P so I now have by assumption that P is true. But since P is true and we also have its antecedent, then its consequent follows by modus ponens so given my assumption of "this conditional is true" then Logic is Awesome so by conditional proof (→Intro), I have now proved P. But then the antecedent "this conditional is true" is in fact true since I have just proved that P is true. Therefore by modus ponens, it follows that:

Logic is Awesome.

WHAT IS GOING ON?

It looks like we just said ‘lets examine a sentence P” where $P \leftrightarrow (P \rightarrow T)$. For any such sentences, both P and T are true This is just like a knight/knave saying: “If I am a knight then T”

The P that we examined “If this sentence is true, then logic is awesome”. This is equivalent to “Either this sentence is false, or Logic is awesome”. So the liar sentence is playing an essential role here. We don’t need to blame conditional proof or anything.

Of course the argument not good, but the conclusion is true:

Logic is Awesome.

Summary of the Course

Friday, 3 December

What is Logic?

What is Logic?

- Reasoning is ubiquitous in every aspect of life. Arguably, the foundation of reasoning is <u>consistency</u>. Consistency is intimately related to the concept of logical consequence.

What is Logic?

- Reasoning is ubiquitous in every aspect of life. Arguably, the foundation of reasoning is consistency. Consistency is intimately related to the concept of logical consequence.
- We say that some proposition P is a logical consequence of a set of propositions S if it is impossible for every proposition in S to be true and P to be false.

What is Logic?

- Reasoning is ubiquitous in every aspect of life. Arguably, the foundation of reasoning is consistency. Consistency is intimately related to the concept of logical consequence.
- We say that some proposition P is a logical consequence of a set of propositions S if it is impossible for every proposition in S to be true and P to be false.
- In one view, Logic is the study of the logical consequence relation. (Or maybe better, deductive logic)

What is Logical Consequence?

What is Logical Consequence?

- But which arguments are valid and what do we mean by “impossible”. What is a possible circumstance?

What is Logical Consequence?

- But which arguments are valid and what do we mean by "impossible". What is a possible circumstance?
- One way to tell the story of the class is to look at a series of ways of getting better and better at formalizing logical consequence and finally, examining fundamental facts about the scope and limits of formal systems themselves.

What is Logical Consequence?

- But which arguments are valid and what do we mean by "impossible". What is a possible circumstance?
- One way to tell the story of the class is to look at a series of ways of getting better and better at formalizing logical consequence and finally, examining fundamental facts about the scope and limits of formal systems themselves.
- Step 1: Develop a formal language for describing the propositions and arguments of interest. We have FOL which is the foundation of all natural and artificial languages that we know of.

FO Languages

FO Languages

- Sentences are true or false in particular circumstances.

FO Languages

- Sentences are true or false in particular circumstances. (?)

FO Languages

- Sentences are true or false in particular circumstances. (?)
- Some sentences are atomic - they don't have any smaller sentences as parts. These have a subject-predicate form.

FO Languages

- Sentences are true or false in particular circumstances. (?)
- Some sentences are atomic - they don't have any smaller sentences as parts. These have a subject-predicate form. (?)

FO Languages

- Sentences are true or false in particular circumstances. (?)
- Some sentences are atomic - they don't have any smaller sentences as parts. These have a subject-predicate form. (?)
- Complex sentences are formed from atomic sentences by recursive rules. These rules involve connectives and logical operators. Many connectives are truth-functional, many operators are quantificational in nature.

First pass at Consequence: Sentential Logic

First pass at Consequence: Sentential Logic

- All FOL sentences have a truth-functional structure. And some arguments are valid just in virtue of this structure.

FIRST PASS AT CONSEQUENCE: SENTENTIAL LOGIC

- All FOL sentences have a truth-functional structure. And some arguments are valid just in virtue of this structure.
- Truth-tables systematically check all tf-possible circumstances. But if we want to do it faster (in some cases) or understand our actual reasoning practices, we develop some kind of proof theory such as a natural deduction method. In our class, we learned $\mathcal{F}$.

First pass at Consequence: Sentential Logic

- All FOL sentences have a truth-functional structure. And some arguments are valid just in virtue of this structure.

- Truth-tables systematically check all tf-possible circumstances. But if we want to do it faster (in some cases) or understand our actual reasoning practices, we develop some kind of proof theory such as a natural deduction method. In our class, we learned $\mathcal{F}$.

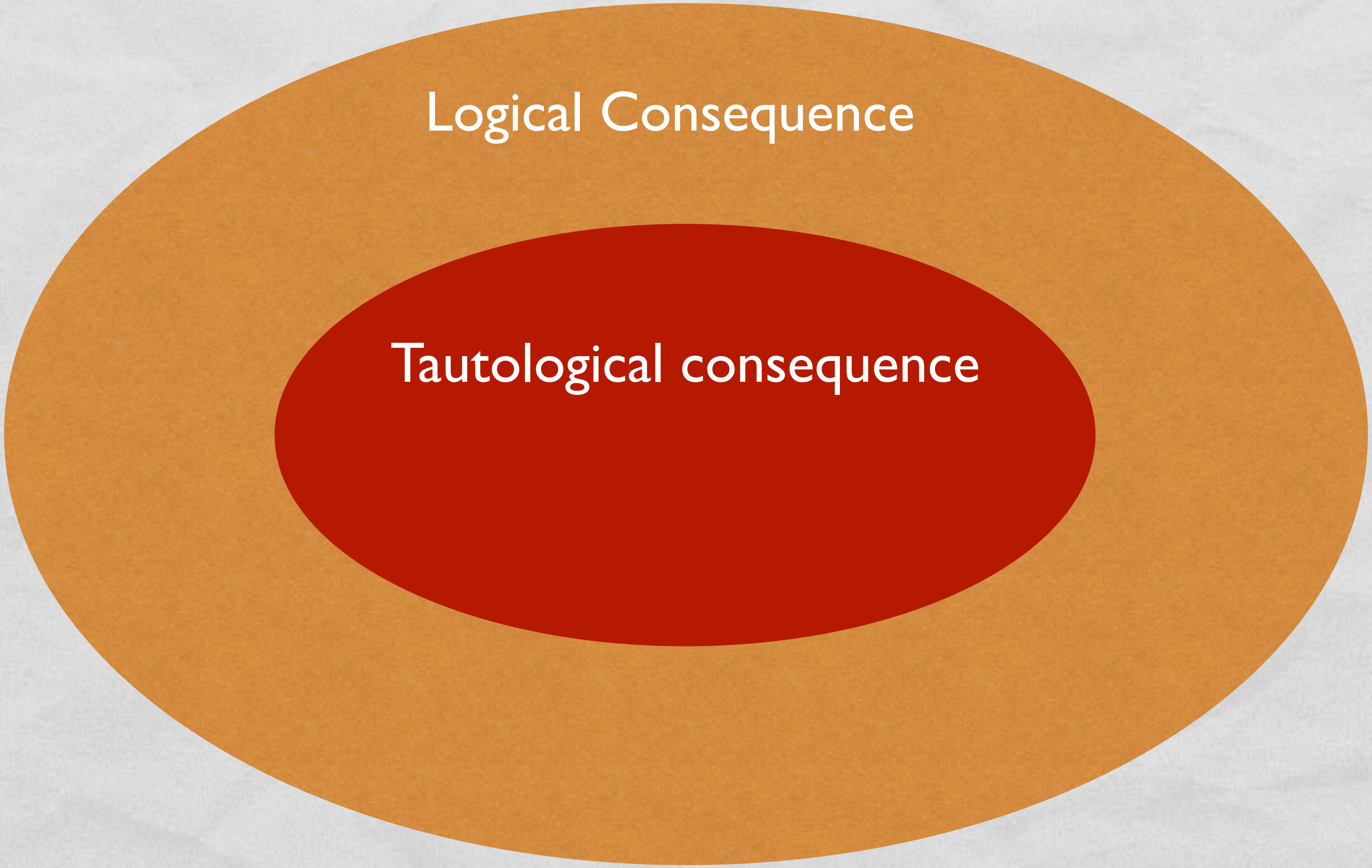
- There are many different systems. It doesn't matter which you use. What matters is that it perfectly matches the truth-tables (soundness and completeness).

Sentential Logic: not good enough

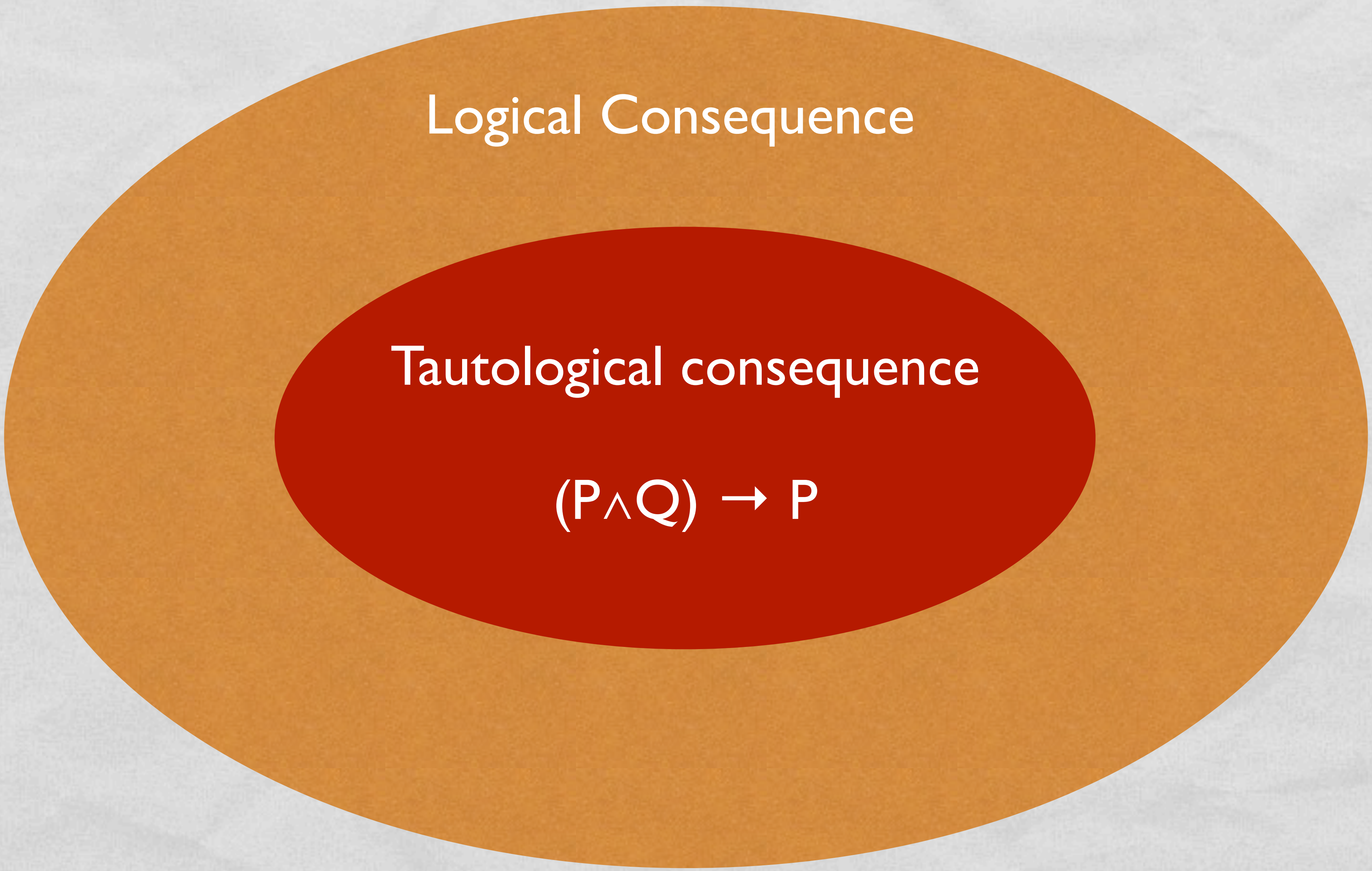
SENTENTIAL LOGIC: NOT GOOD ENOUGH

- But sentential logic is not good enough. There are clearly valid arguments which have invalid tf-forms.

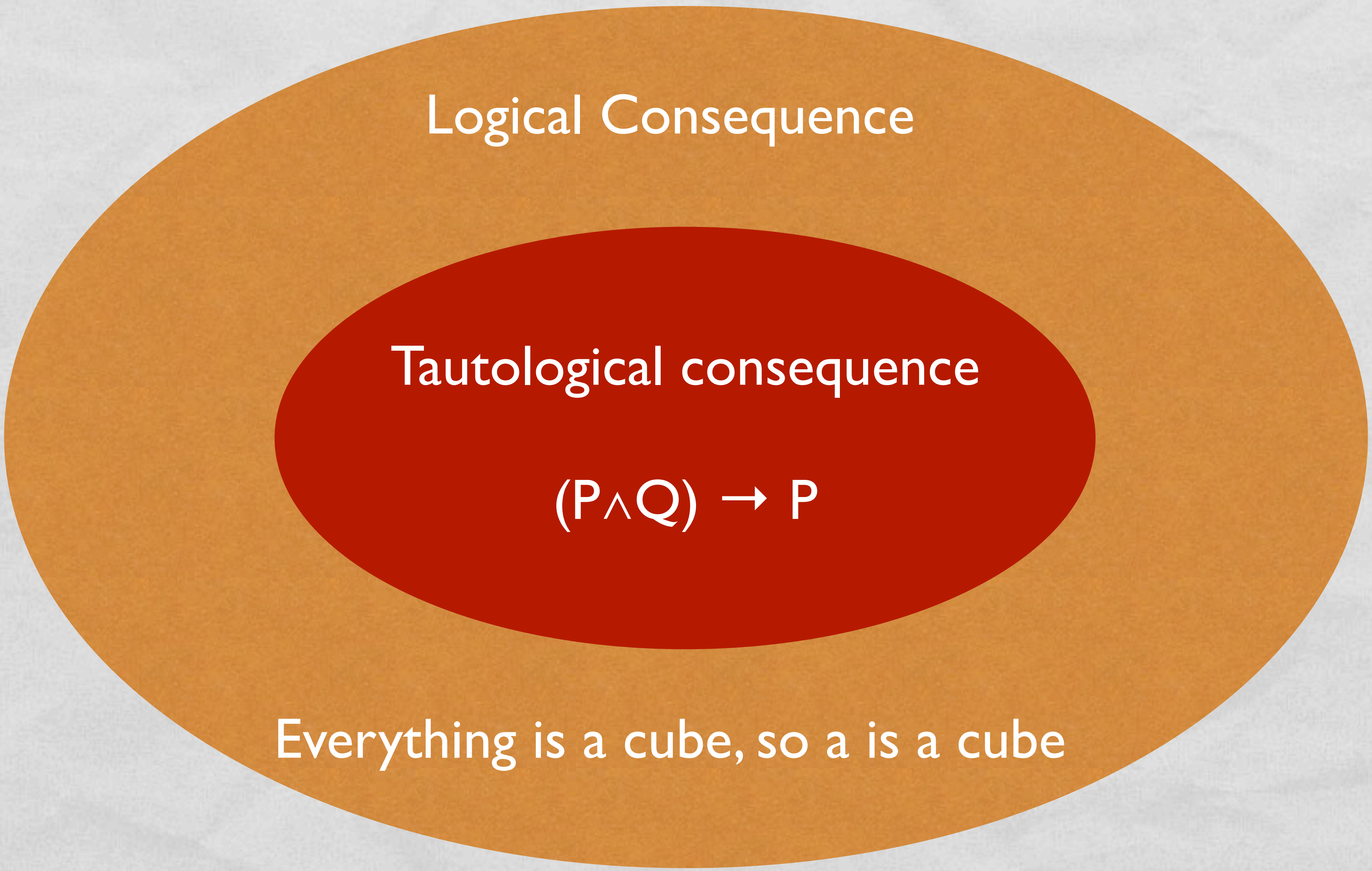
Tautological versus Logical



Tautological versus Logical



Tautological versus Logical



SENTENTIAL LOGIC: NOT GOOD ENOUGH

- But sentential logic is not good enough. There are clearly valid arguments which have invalid tf-forms.

SENTENTIAL LOGIC: NOT GOOD ENOUGH

- But sentential logic is not good enough. There are clearly valid arguments which have invalid tf-forms.
- The next step is to look at the quantificational structure. Many types of quantification are equivalent to some combination of ‘all’ and ‘some’.

SENTENTIAL LOGIC: NOT GOOD ENOUGH

- But sentential logic is not good enough. There are clearly valid arguments which have invalid tf-forms.
- The next step is to look at the quantificational structure. Many types of quantification are equivalent to some combination of 'all' and 'some'.
- We are still concerned with consistency and 'all possible circumstances'. But now, a possible circumstance is there could be any number of things and any particular predicate could be true of any number of them, relation true of any pair, etc.

Second Pass: Predicate Logic

Second Pass: Predicate Logic

- Some arguments are valid just because of their quantificational structure. Here, we say if all the premises are true regardless of whatever objects satisfy the predicates in question, then the conclusion is a first order consequence of the premises. (exception - we allow the meaning of the identity predicate)

Second Pass: Predicate Logic

- Some arguments are valid just because of their quantificational structure. Here, we say if all the premises are true regardless of whatever objects satisfy the predicates in question, then the conclusion is a first order consequence of the premises. (exception - we allow the meaning of the identity predicate)
- Sometimes 'logical consequence' is just defined this way as FO consequence. (Similarly, logical consistency, possibility, etc.) Here the idea is that the quantifiers, connectives, and identity are logical symbols, the predicates aren't. Things can be true in our world because of facts about our world, or they might be logically true because of FOL.

Second Pass: Predicate Logic

Second Pass: Predicate Logic

- We have the same semantics vs. syntax descriptions of consequence.

Second Pass: Predicate Logic

- We have the same semantics vs. syntax descriptions of consequence.
- Semantics as basic says consequence means any model that makes all the premises true also makes the conclusion true.

Second Pass: Predicate Logic

- We have the same semantics vs. syntax descriptions of consequence.
- Semantics as basic says consequence means any model that makes all the premises true also makes the conclusion true.
- But we could also say that validity is provability in some proof system - like $\mathcal{F}$. Historically, we had to agree what counted as FO valid and we first had a proof system which seemed to capture this. Now we have model theory with a set-theoretic description of what a model is that matches (sound + complete) the proof rules.

Second Pass: Predicate Logic

Second Pass: Predicate Logic

- Once we get to predicate logic, we can start to make a serious study of the properties of the formal system (and all possible formal systems).

Second Pass: Predicate Logic

- Once we get to predicate logic, we can start to make a serious study of the properties of the formal system (and all possible formal systems).

- For example, sentential logic is decidable. There is a single algorithm (eg truth tables) that will correctly decide the truth of any question like 'is P a tautology' in finite time.

Second Pass: Predicate Logic

- Once we get to predicate logic, we can start to make a serious study of the properties of the formal system (and all possible formal systems).

- For example, sentential logic is <u>decidable</u>. There is a single algorithm (eg truth tables) that will correctly decide the truth of any question like 'is P a tautology' in finite time.

- However, with a language with at least one two (or more) place predicate, theoremhood (and consistency, entailment, etc.) are undecidable. There is no algorithm which will answer all questions of the form "Is P an FO validity?"

FOL WITH MEANING POSTULATES

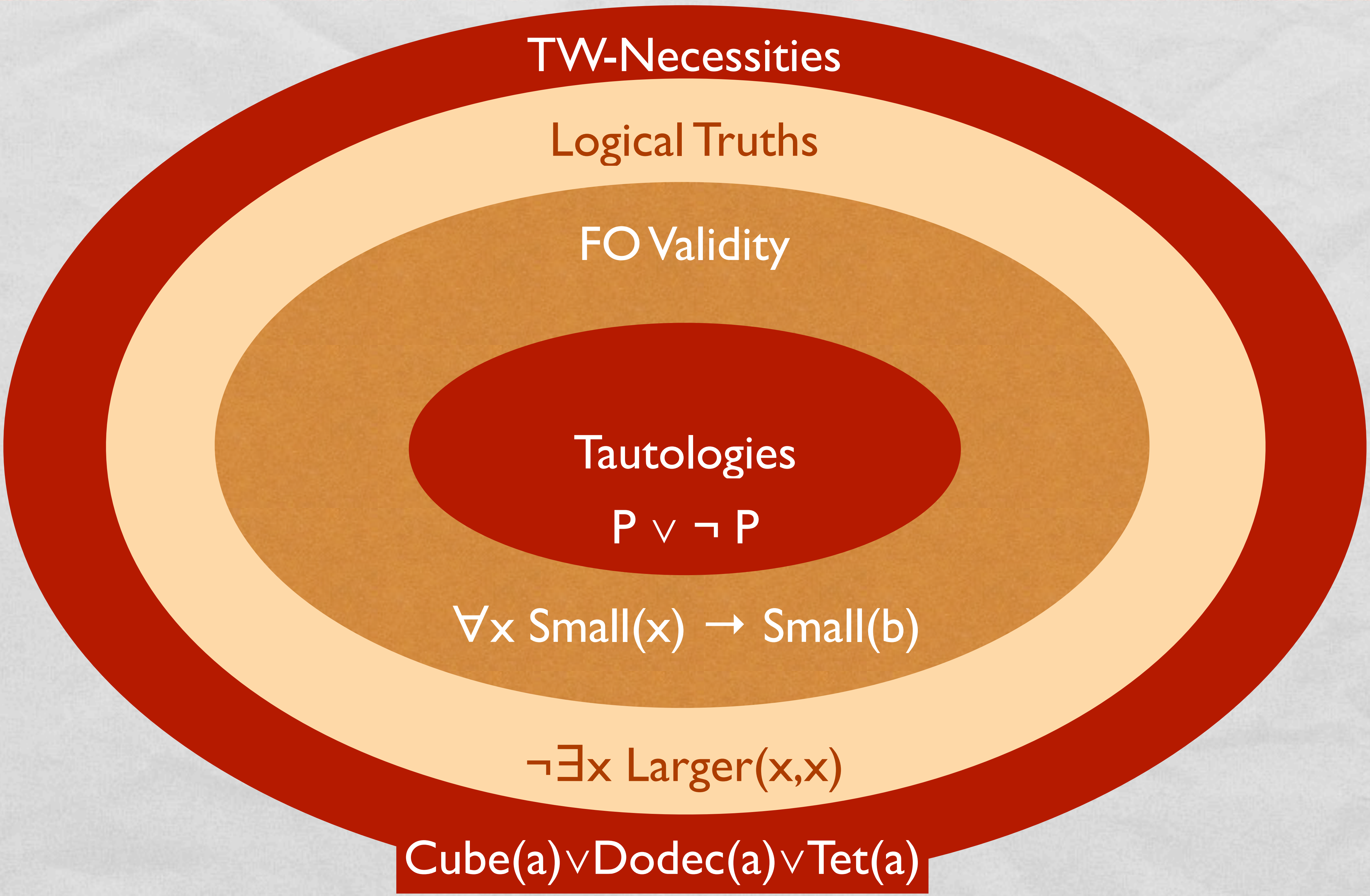
FOL WITH MEANING POSTULATES

- There are still arguments that are valid but not FO valid. For example, everything is the same shape as itself.

FOL WITH MEANING POSTULATES

- There are still arguments that are valid but not FO valid. For example, everything is the same shape as itself.
- FO validity captures consequence in virtue of the logical terms. But some arguments are valid in virtue partially of the meaning of the predicates. Our book thinks of these as logical truths. You don't have to, but you should have something to say about them.

First-Order Validity and Consequence



FOL WITH MEANING POSTULATES

FOL WITH MEANING POSTULATES

- There are still arguments that are valid but not FO valid. For example, everything is the same shape as itself.

FOL WITH MEANING POSTULATES

- There are still arguments that are valid but not FO valid. For example, everything is the same shape as itself.
- One description of this is that it is an 'analytic truth'. It is sometimes thought that FOL can capture the (logical?) truth of this sentence by adding meaning postulates, definitions, axioms, or whatever to transform this into an FO-validity.

FOL WITH MEANING POSTULATES

- There are still arguments that are valid but not FO valid. For example, everything is the same shape as itself.
- One description of this is that it is an 'analytic truth'. It is sometimes thought that FOL can capture the (logical?) truth of this sentence by adding meaning postulates, definitions, axioms, or whatever to transform this into an FO-validity.
- Personally, I think this is impossible to always do. But I am not sure these are logical consequences anyway.

DID WE GET THEM ALL?

DID WE GET THEM ALL?

- We didn't spend any time on the limits of FOL. But there are many.

Did we get them all?

- We didn't spend any time on the limits of FOL. But there are many.
- For example, many kinds of quantification can't actually be captured by all and some (chap 14). Many, most, at least 2/3s of, a finite number of, etc. can't be captured perfectly in FOL.

Did we get them all?

- We didn't spend any time on the limits of FOL. But there are many.
- For example, many kinds of quantification can't actually be captured by all and some (chap 14). Many, most, at least 2/3s of, a finite number of, etc. can't be captured perfectly in FOL.
- Valid argument: More than half of the Ps are Qs. So there are more things that are $P \wedge Q$ than are $P \wedge \neg Q$.

Did we get them all?

- We didn't spend any time on the limits of FOL. But there are many.
- For example, many kinds of quantification can't actually be captured by all and some (chap 14). Many, most, at least 2/3s of, a finite number of, etc. can't be captured perfectly in FOL.
- Valid argument: More than half of the Ps are Qs. So there are more things that are $P \wedge Q$ than are $P \wedge \neg Q$.
- This is expressible with set theory, but this seems to imply too much (like that there are sets!)

Higher Order Logic

Higher Order Logic

- In FOL, we quantify over objects in the domain. All means ‘all elements of the domain of the model’.

Higher Order Logic

- In FOL, we quantify over objects in the domain. All means 'all elements of the domain of the model'.
- Notice that the following are all theorems:

  $\forall x[P(x) \vee \neg P(x)]$

  $\forall x[Q(x) \vee \neg Q(x)]$

  $\forall x[R(x) \vee \neg R(x)]$, etc.

Higher Order Logic

- In FOL, we quantify over objects in the domain. All means 'all elements of the domain of the model'.
- Notice that the following are all theorems:

  $\forall x[P(x) \vee \neg P(x)]$

  $\forall x[Q(x) \vee \neg Q(x)]$

  $\forall x[R(x) \vee \neg R(x)]$, etc.
- In FOL we can't express the claim that 'for any predicate, that form is a theorem'. We want to say $\forall P\ \forall x[P(x) \vee \neg P(x)]$ where the first quantifier ranges over predicates, not objects.

Higher Order Logic

Higher Order Logic

- In FOL we can't express the claim that 'for any predicate, that form is a theorem'. We want to say $\forall P\ \forall x[P(x) \vee \neg P(x)]$ where the first quantifier ranges over predicates, not objects.

Higher Order Logic

- In FOL we can't express the claim that 'for any predicate, that form is a theorem'. We want to say $\forall P\ \forall x[P(x) \vee \neg P(x)]$ where the first quantifier ranges over predicates, not objects.

- This is called second order logic because predicates can be represented as subsets of the domain. The domain of the second order quantifiers is the powerset of the first order quantifiers domain.

Higher Order Logic

- In FOL we can't express the claim that 'for any predicate, that form is a theorem'. We want to say $\forall P\ \forall x[P(x) \vee \neg P(x)]$ where the first quantifier ranges over predicates, not objects.

- This is called second order logic because predicates can be represented as subsets of the domain. The domain of the second order quantifiers is the powerset of the first order quantifiers domain.

- You can 'model' higher order logic in set theory and prove things about it, but you can't literally express these sentences (in my opinion).

Logic and Avoiding Paradox

Logic and Avoiding Paradox

- It seems clear to me that we can use logic to talk about possible solutions to things like the liar paradox ('This sentence is false').

Logic and Avoiding Paradox

- It seems clear to me that we can use logic to talk about possible solutions to things like the liar paradox ('This sentence is false').
- Tarski's Undefinability Theorem says that powerful formal theories (like FOL) can't have their own truth predicate. But this doesn't mean that there aren't logical facts about truth!

Logic and Avoiding Paradox

- It seems clear to me that we can use logic to talk about possible solutions to things like the liar paradox ('This sentence is false').
- Tarski's Undefinability Theorem says that powerful formal theories (like FOL) can't have their own truth predicate. But this doesn't mean that there aren't logical facts about truth!
- You can say truth isn't part of 'mathematical logic' or something, but that just means you think mathematical logic is limited too. Real logic is not limited and is totally universal and fundamental.

Logic and Avoiding Paradox

- It seems clear to me that we can use logic to talk about possible solutions to things like the liar paradox ('This sentence is false').
- Tarski's Undefinability Theorem says that powerful formal theories (like FOL) can't have their own truth predicate. But this doesn't mean that there aren't logical facts about truth!
- You can say truth isn't part of 'mathematical logic' or something, but that just means you think mathematical logic is limited too. Real logic is not limited and is totally universal and fundamental.

You can't say that 'logic doesn't apply here'. Ever.