Fun with Infinity

The Hilbert Hotel is very large. It has an infinite number of rooms - one for each natural number.

It is possible (even in theory) for the Hilbert Hotel to be full? What if full means:

1) Each room is occupied

or

2) If anyone else came, the hotel could not accommodate them

Set Theory

Monday, 29 November

Formal Semantics

- An interpretation (or <u>structure</u> in LPL chap 18) has:
  - A set of objects as the domain
  - A set of objects for each one place predicate (the set of things that satisfy the predicate)
  - A set of ordered pairs for each two place predicate [n-tuples for the n-place predicates]
  - A function which maps names into objects in the domain
  - Assigns a function (domain to domain) for each function in the language

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Give a model of the following set of sentences:

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$\exists x \exists y(R(x,y) \wedge S(y,x))$

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$\forall x \forall y(R(x,y) \rightarrow R(y,x))$

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Domain: $\{1,2\}$

$R(x,y)$: $\{\langle 1,2 \rangle\}$

$S(x,y)$: $\{\langle 2,1 \rangle\}$

Takes care of P1

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Takes care of P1, P2

Domain: $\{1,2,3,4\}$

$R(x,y)$: $\{<1,2>\}$

$S(x,y)$: $\{<2,1>\}$

$B(x,y,z)$: $\{<1,2,3>\}$

$D(x,y,z)$: $\{<1,2,3,4>\}$

a: 1

b: 2

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Added for P3

Set Theory Notation and Logic

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Of course there is this mathematical theory - set theory - written in FOL where there are axioms giving the formal definition of set. In set theory there is one 2-place predicate E(x,y) and $a \in b$ is just shorthand for E(a,b)

Set Theory Notation and Logic

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We want sets to just be defined by their members

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$\{1,2,3\} = \{2,3,1\} = \{1,2,2,3,3,1,2,3\}$ because:

$1 \in$ the set

$2 \in$ the set

$3 \in$ the set

Nothing else is in the set

Set Theory Notation and Logic

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- 1 $\in$ the set
- 2 $\in$ the set
- 3 $\in$ the set

Nothing else is in the set

To formalize this we have the first axiom: Extentionality

$$\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y]$$

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{x| x is a person in this room}

{{x,y} | x,y live in Ithaca and x,y are married to each other}

{<x,y> | x is a book written by Charles Darwin and y is a pen in my backpack}

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This leads to some paradoxes e.g. The set of all truths

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Really, this is an <u>axiom scheme</u> with an infinite number of axioms:

$\exists y \forall x[x \in y \leftrightarrow x \neq x]$ - the empty set

$\exists y \forall x[x \in y \leftrightarrow x=3]$ - the singleton set {3}

$\exists y \forall x[x \in y \leftrightarrow (x \in a \vee x \in b)]$ - the union of a,b

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Laws of Boolean Algebras

DeMorgan's Laws

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$$C \setminus (A \cup B) = C \setminus A \cap C \setminus B$$

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Distribution Laws

Laws of Boolean Algebras

DeMorgan's Laws

$$C \setminus (A \cup B) = C \setminus A \cap C \setminus B$$

$$(A \cup B)^c = A^c \cap B^c$$

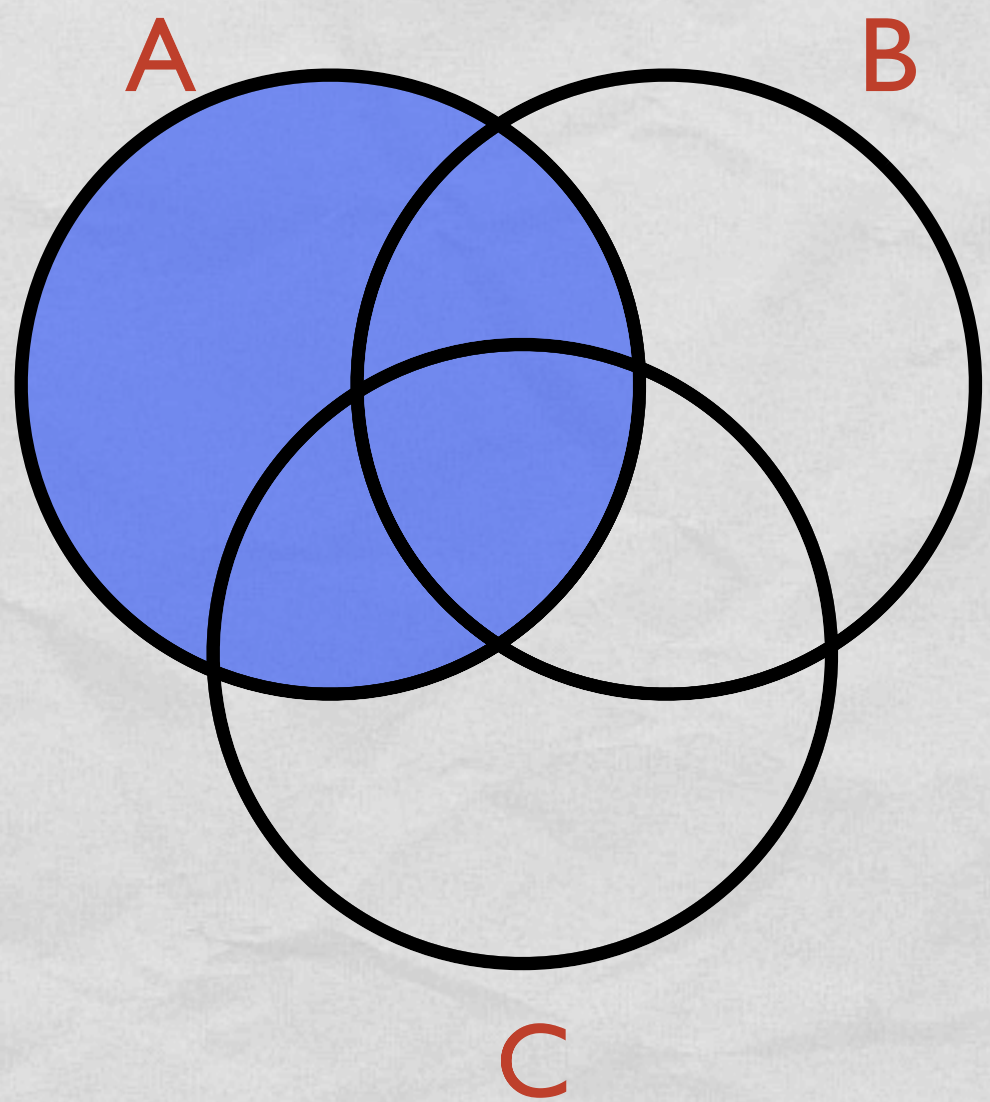
Distribution Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Venn Diagrams

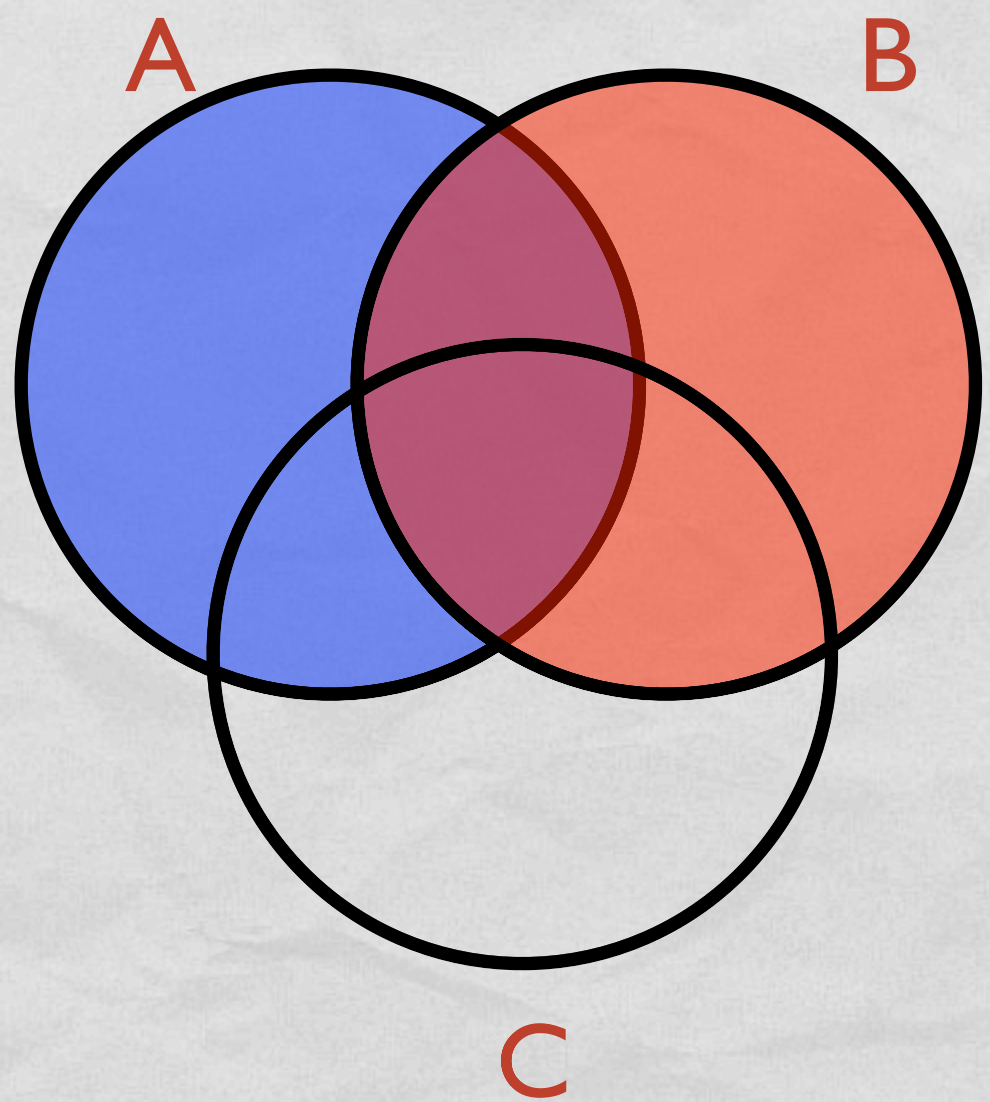
Everything in A = blue



Venn Diagrams

Everything in A = blue

Everything in B = red



Venn Diagrams

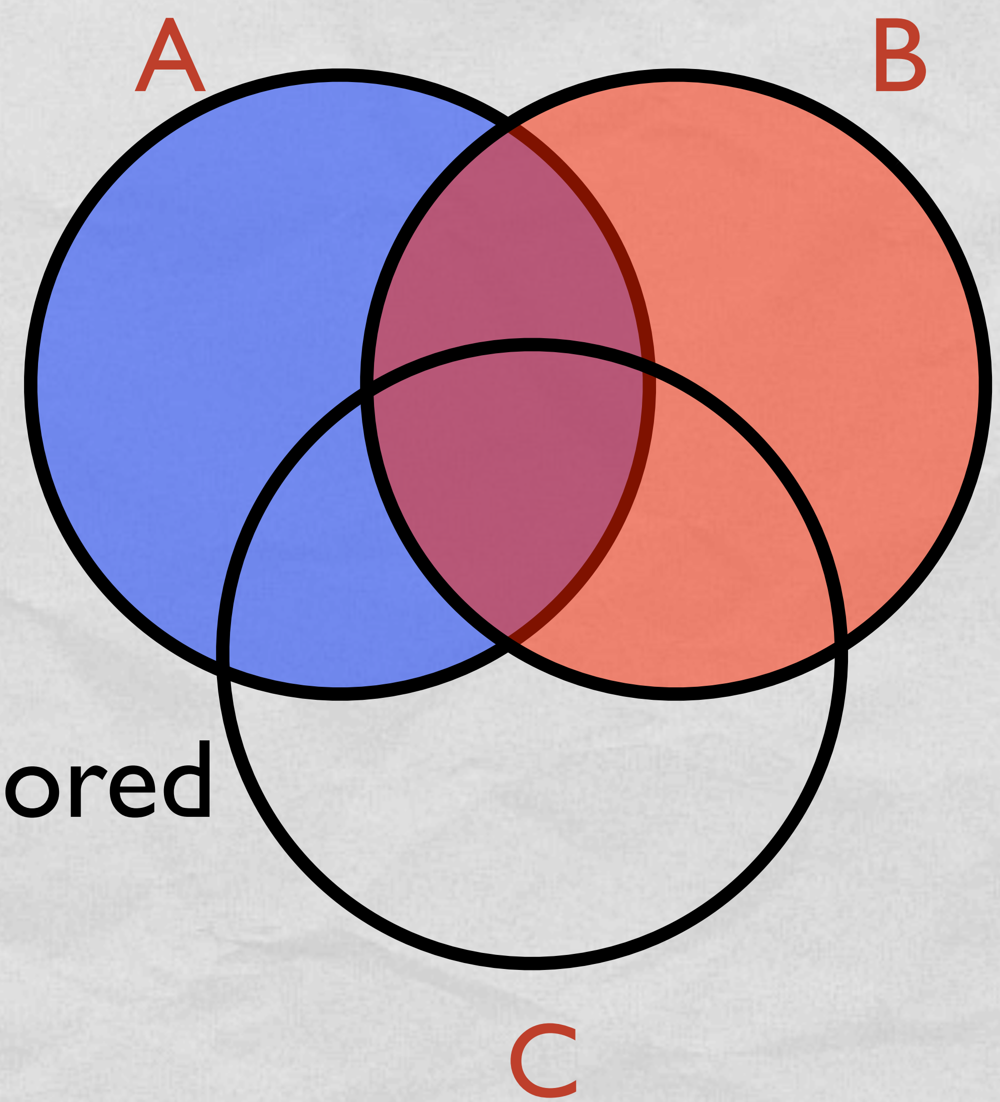
Everything in A = blue

Everything in B = red

A ∪ B = colored

A ∩ B = purple

C∖(A ∪ B) = in C but not colored



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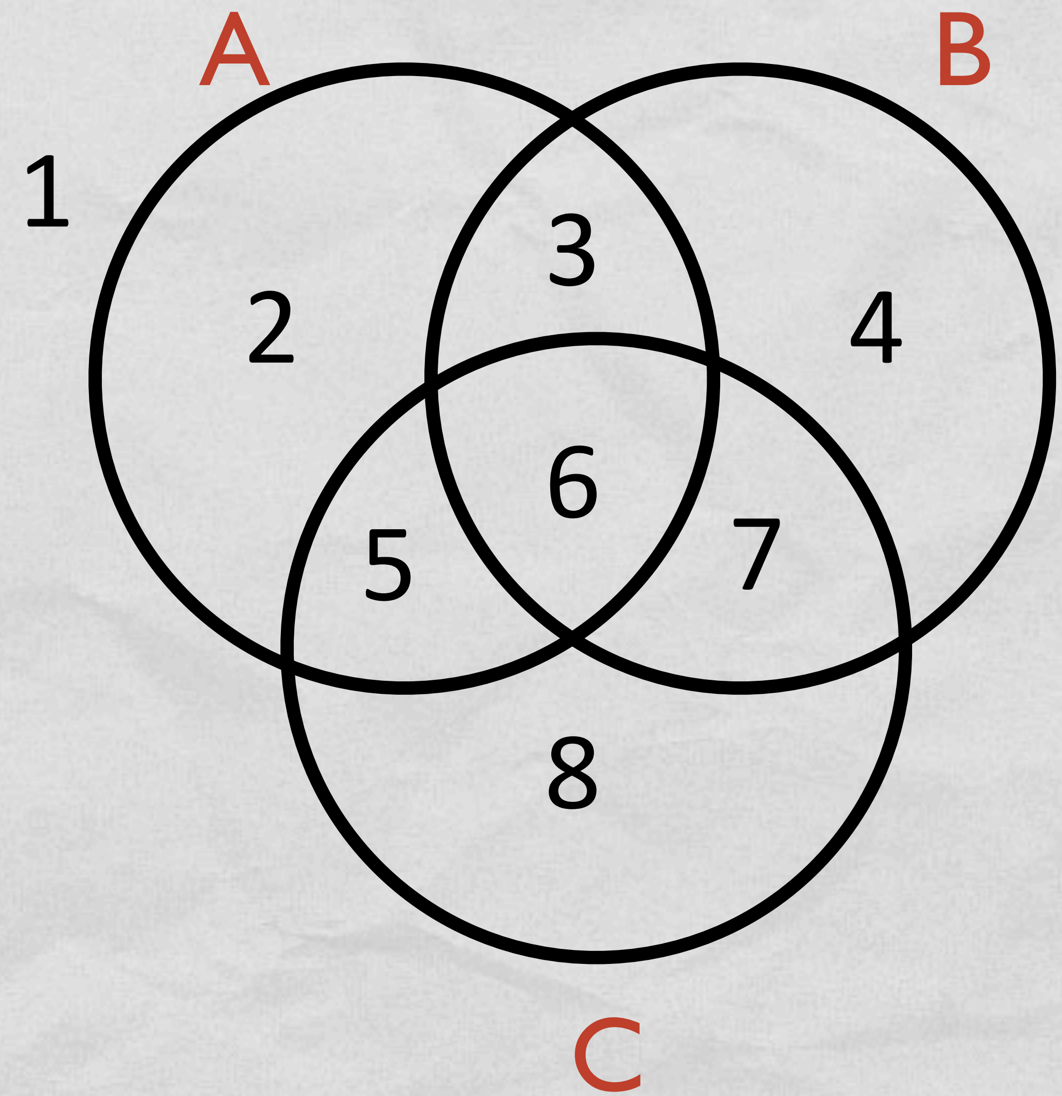
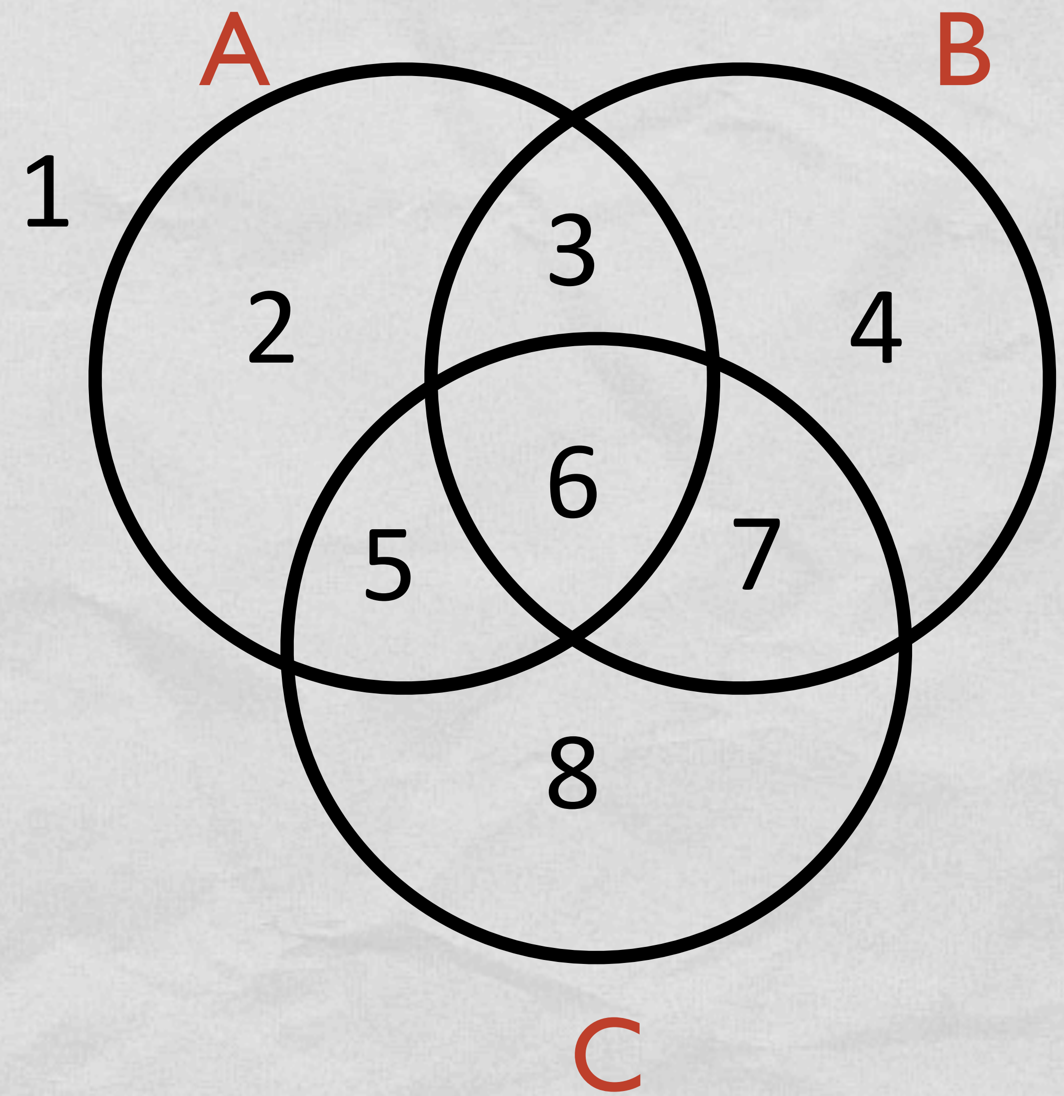


Venn Diagrams

region 6 = $A \cap B \cap C$

region 3 = $A \cap B \cap C^c$



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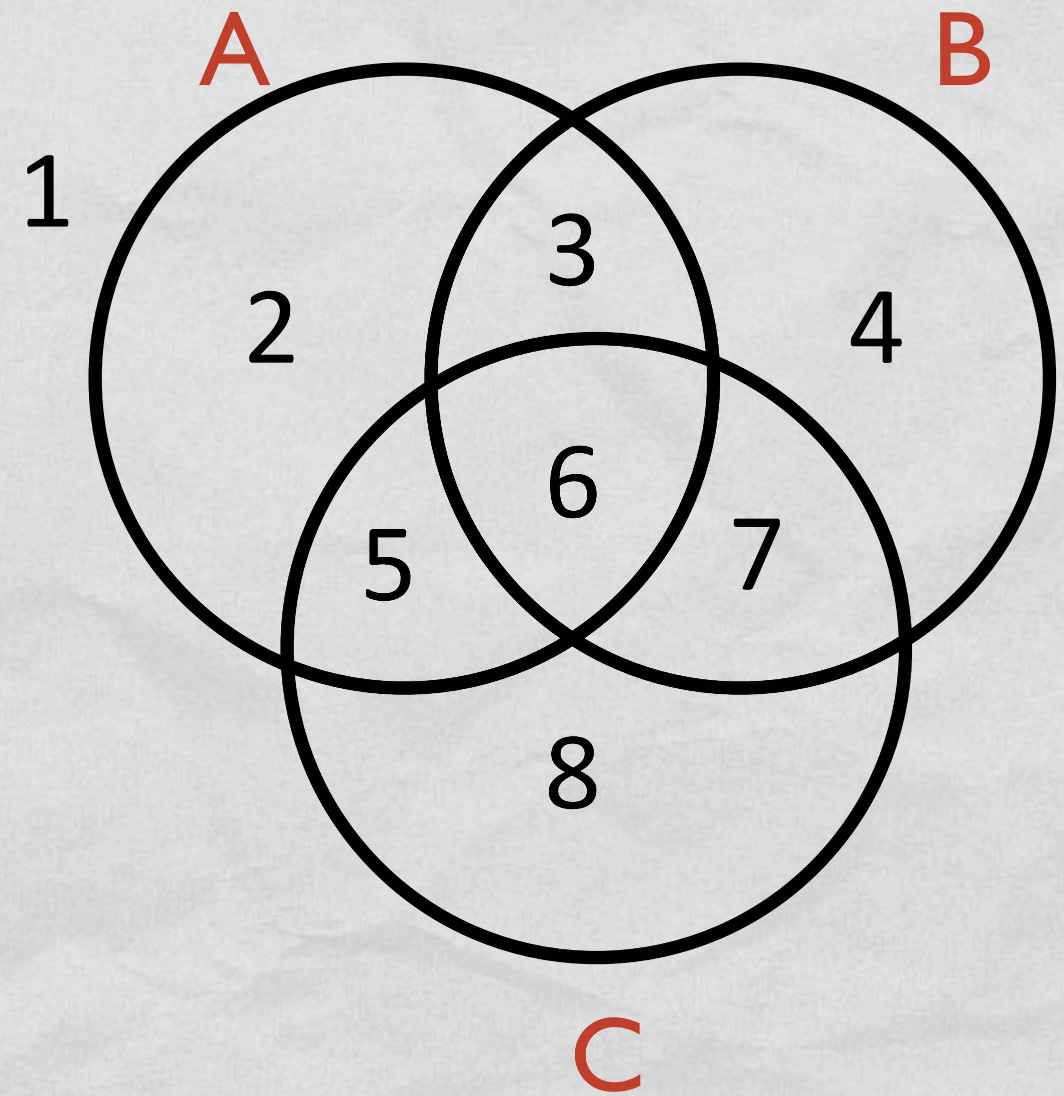

$A = 2+3+5+6$

$A \cap B = 2+3+5+6 \cap 3+4+6+7$

$= 3+6$

VENN DIAGRAMS

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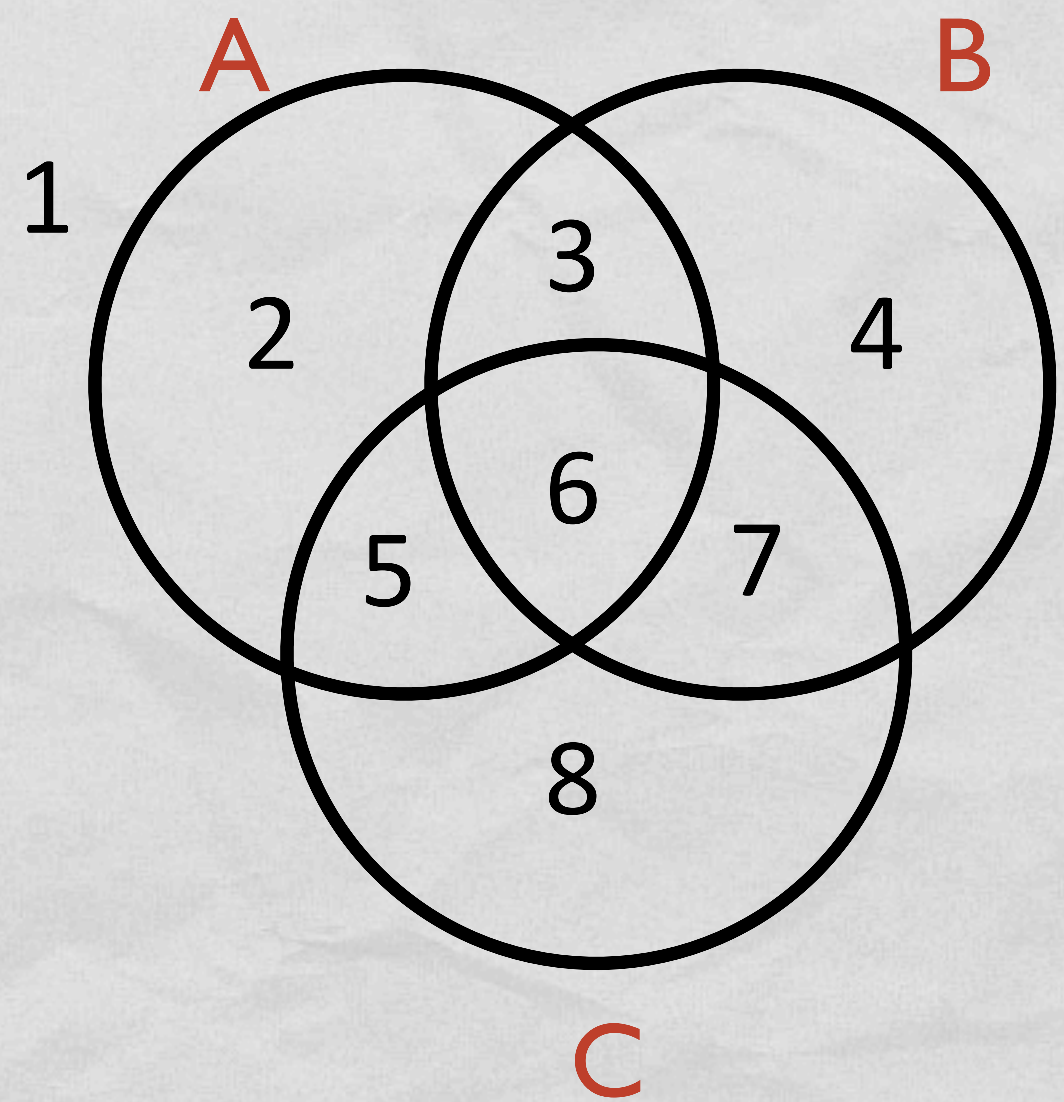
A=2+3+5+6

$A \cap B$ = 2+3+5+6 ∩ 3+4+6+7

= 3+6

region 5+6+7 =

$C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$



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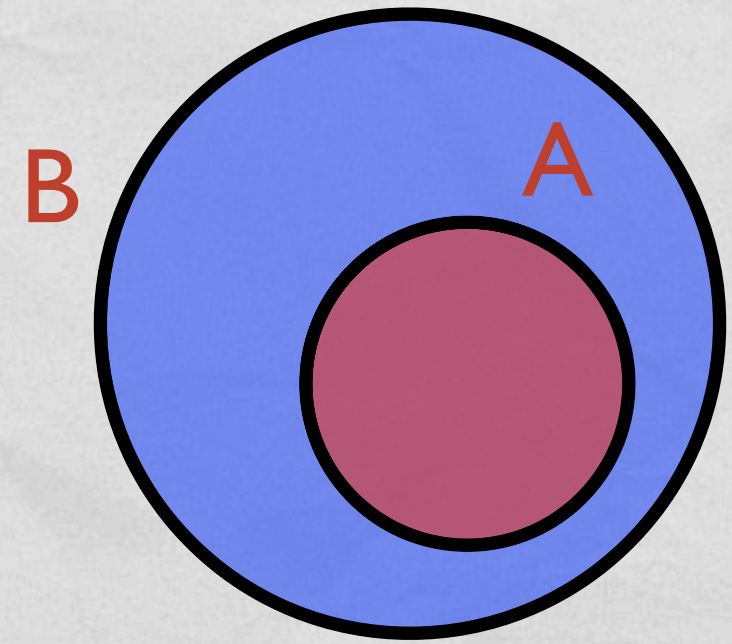

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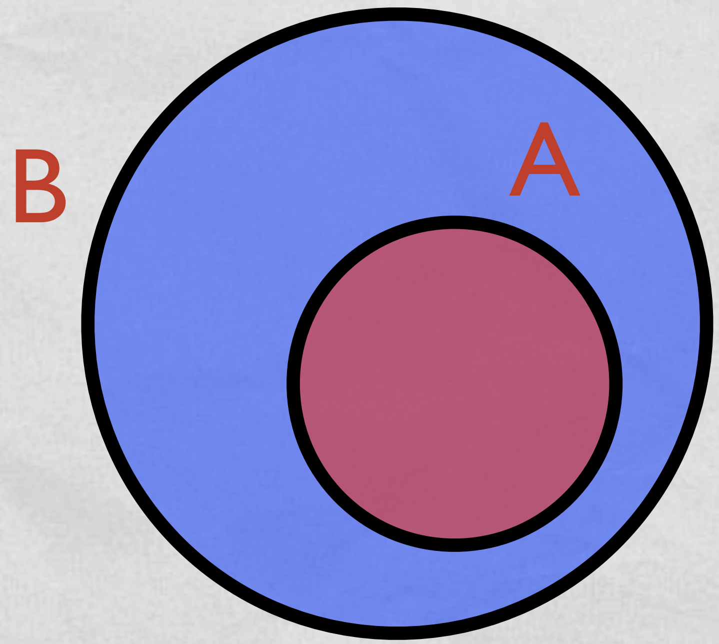
$A \subseteq B =_{def} \forall x(x \in A \rightarrow x \in B)$



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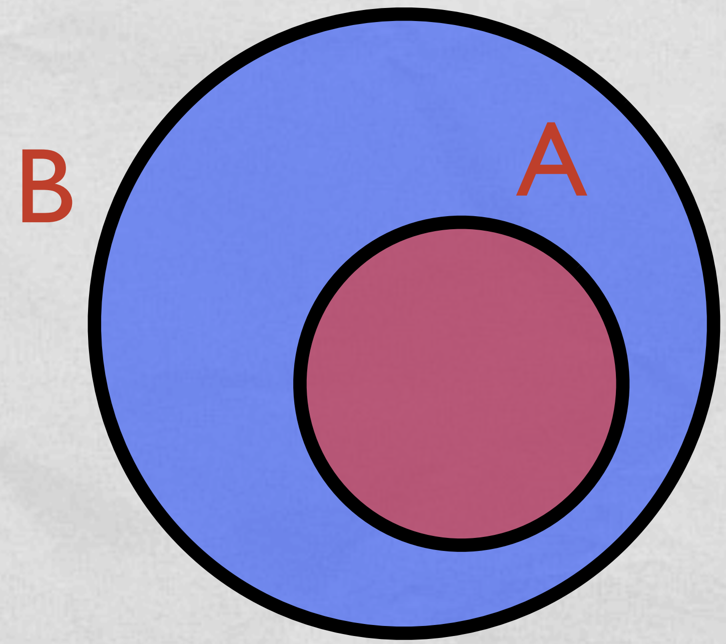

$A \subsetneq B =_{def} A \subseteq B \wedge A \neq B$

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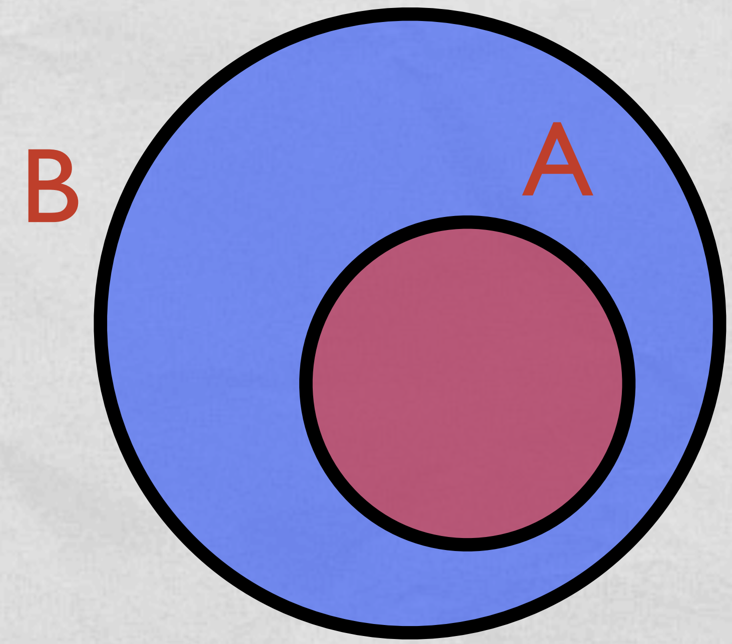

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A is a "proper subset" of B

$\vdash (A \subseteq B \wedge B \subseteq A) \rightarrow A=B$

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Membership isn't (generally)

$(1 \in \{1,2\}, \{1,2\} \in \{\{1,2\}, \{2,3\}\}$

but $1 \notin \{\{1,2\}, \{2,3\}\}$

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FACT: The number of elements in $\wp(A)$ = "size" of $\wp(A)$ = 2 raised to the power of the size of A.

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$\vdash \forall x \ |\wp(x)| = 2^{|x|}$

and yes, $\vdash \forall x \ (x < 2^x)$ even for infinite x

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-Cantor's Paradox-

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From these two we can prove $U = \wp(U)$

-- but we just showed that $\vdash |\wp(U)| < |U|$

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$R \in R \leftrightarrow R \notin R$

$\vdash \neg \exists y \forall x[x \in y \leftrightarrow x \notin x]$ by Russell's argument

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So from a set z, you can shrink it down.

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- Powerset (we can make the powerset)
- Axiom of Infinity (the natural numbers are a set)
- +Replacement, Foundation, Choice make ZFC Set Theory