EUCLID'S POSTULATES (AXIOMS) DOES 5) FOLLOW FROM THE OTHERS?

1) Can draw a straight line between two points

2) Can extend a straight line (make it longer)

3) Can construct a circle with given center and radius

4) All right angles are equal to each other

5) Parallel Postulate (equivalent to): Through a point not on a given straight line, at most one line can be drawn that never meets the given line

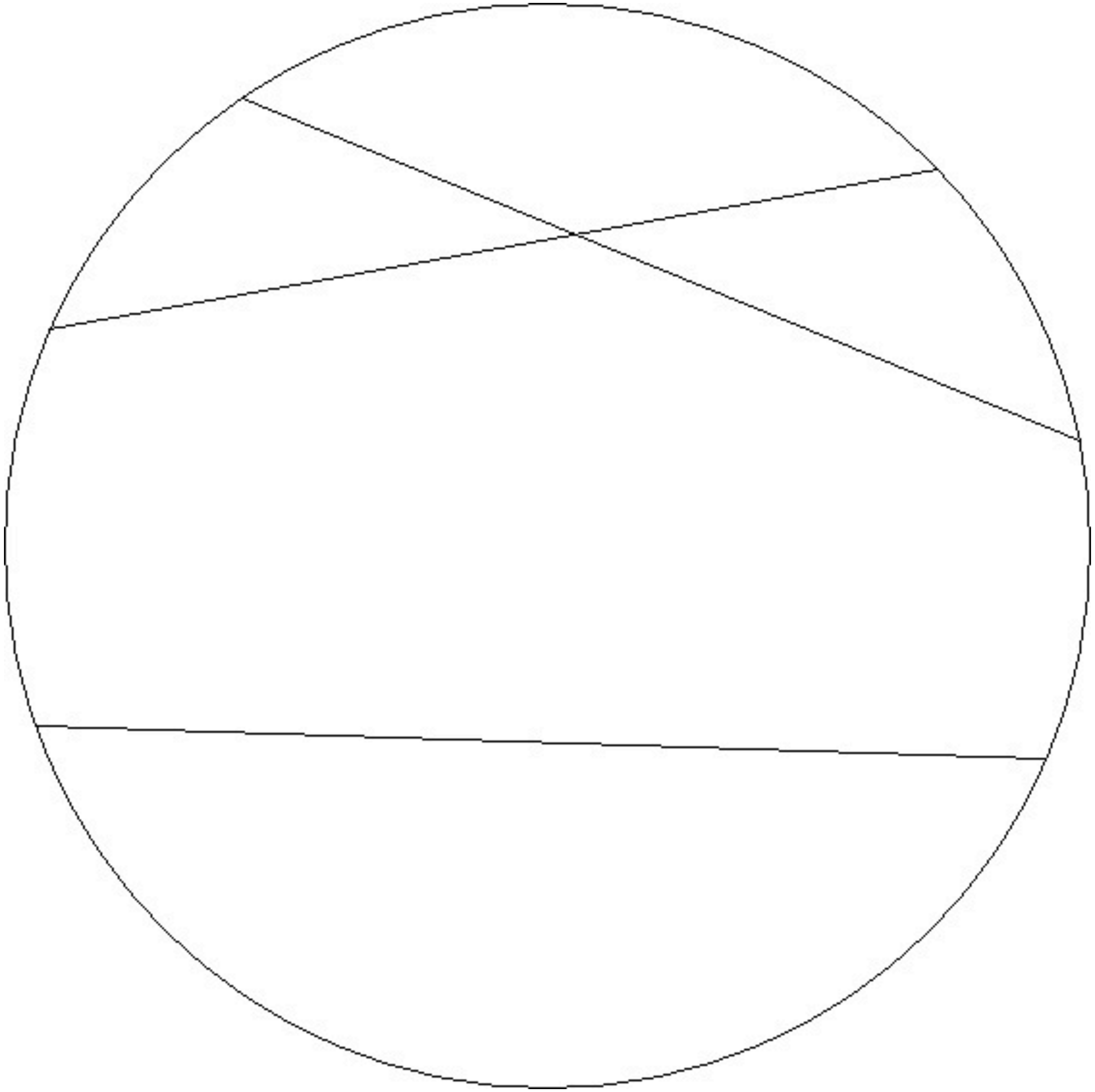
NO
AXIOMS ARE INDEPENDENT

Trying to show that the parallel postulate follows from the others is one of the most famous goals in the history of mathematics.

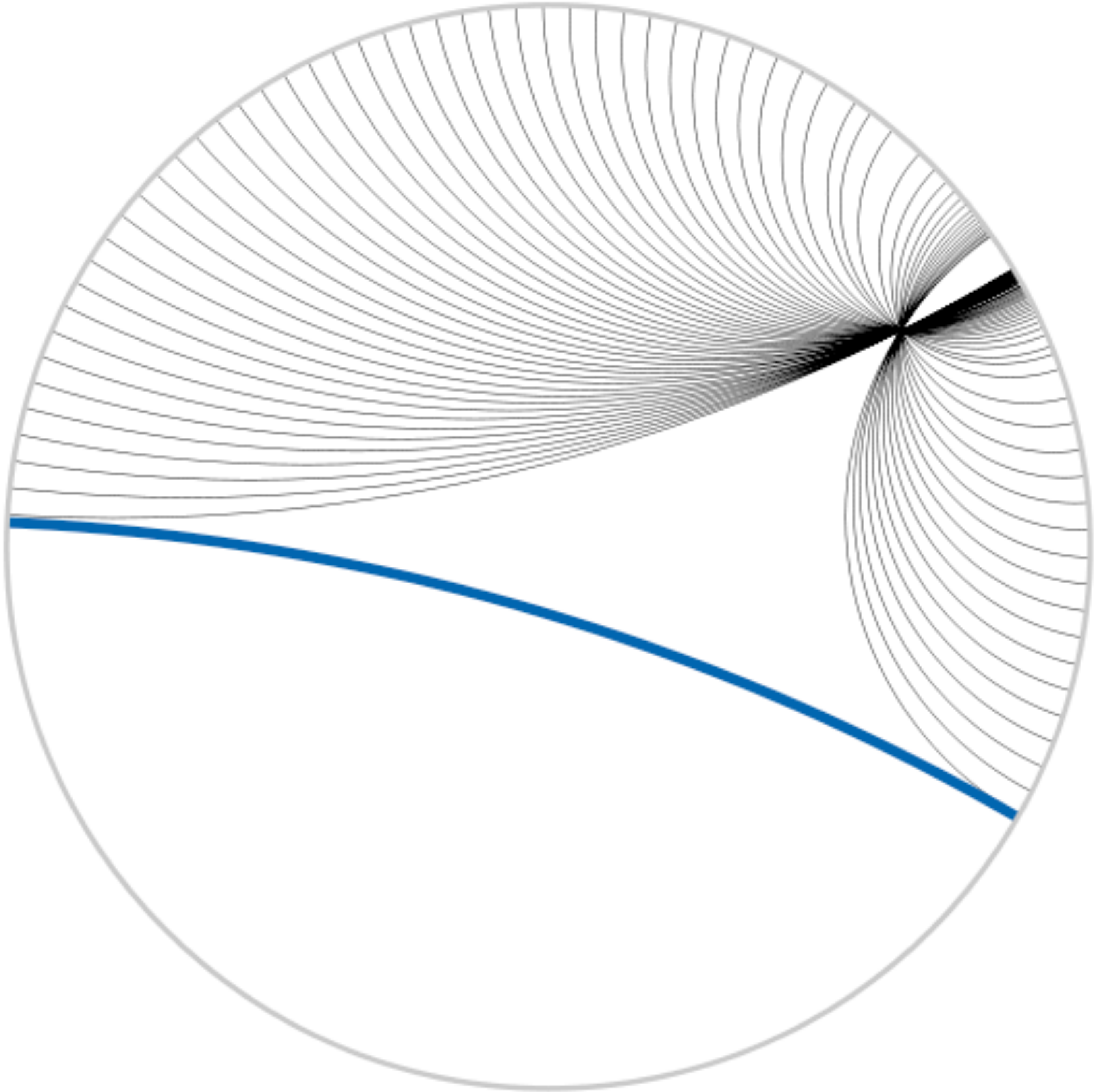
It sure looks like the axioms are all consistent - take “ordinary” [Euclidean] space.

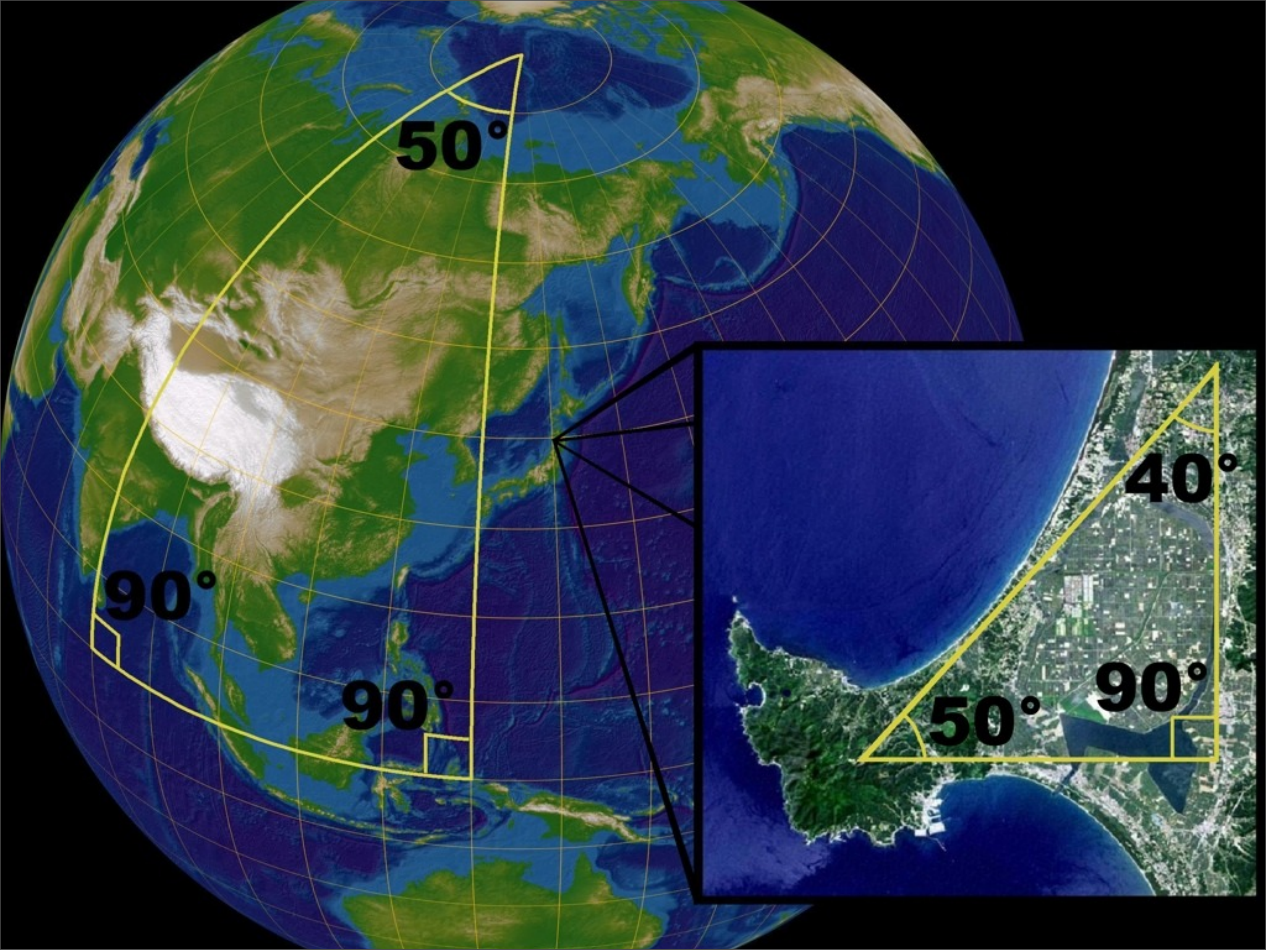
What happens if the parallel postulate is false? Think reductio - this gets you non-Euclidean geometries.

Klein model - “space” is the interior of a circle. “Straight lines” are chords. “Angles” look distorted.

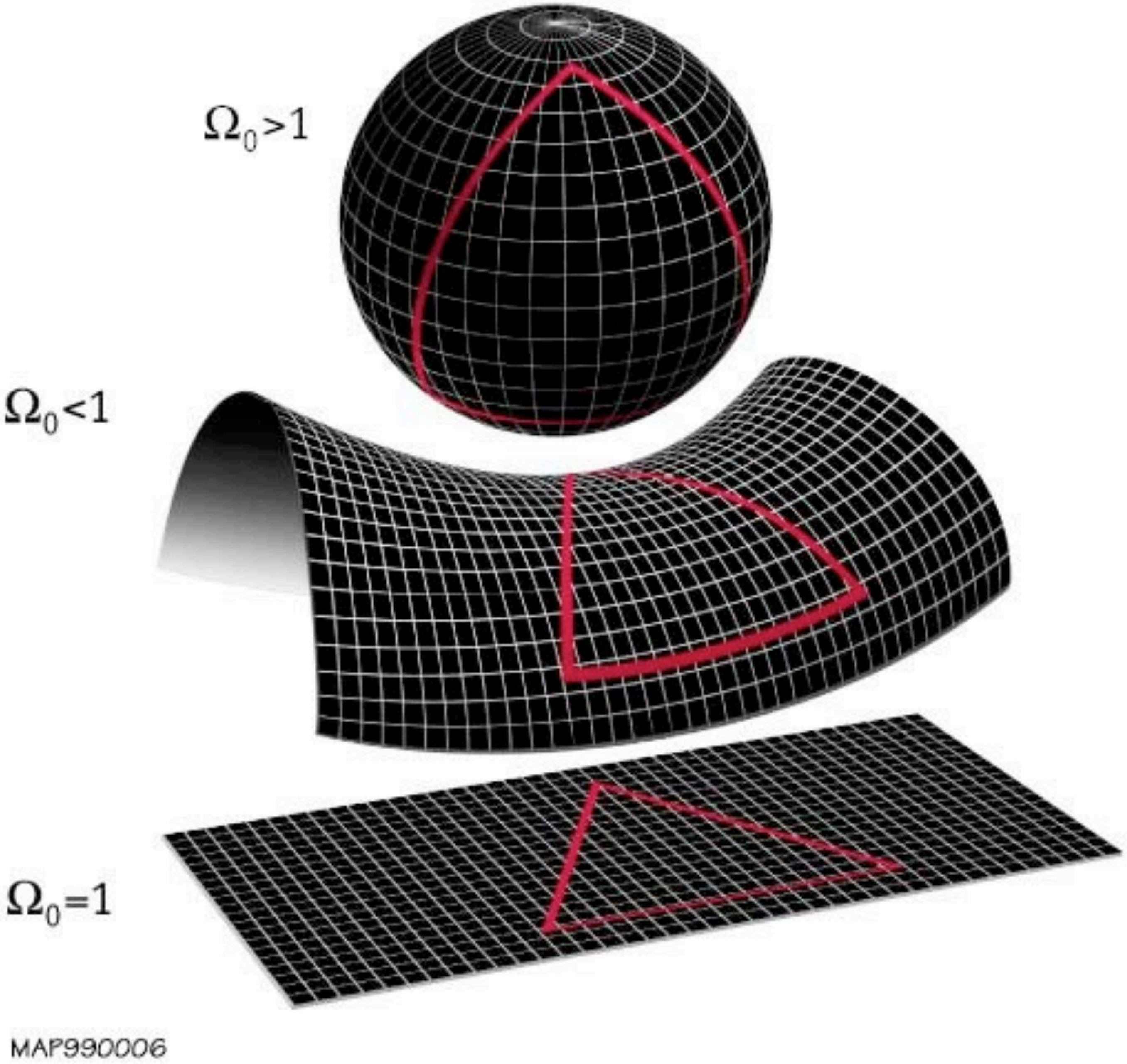


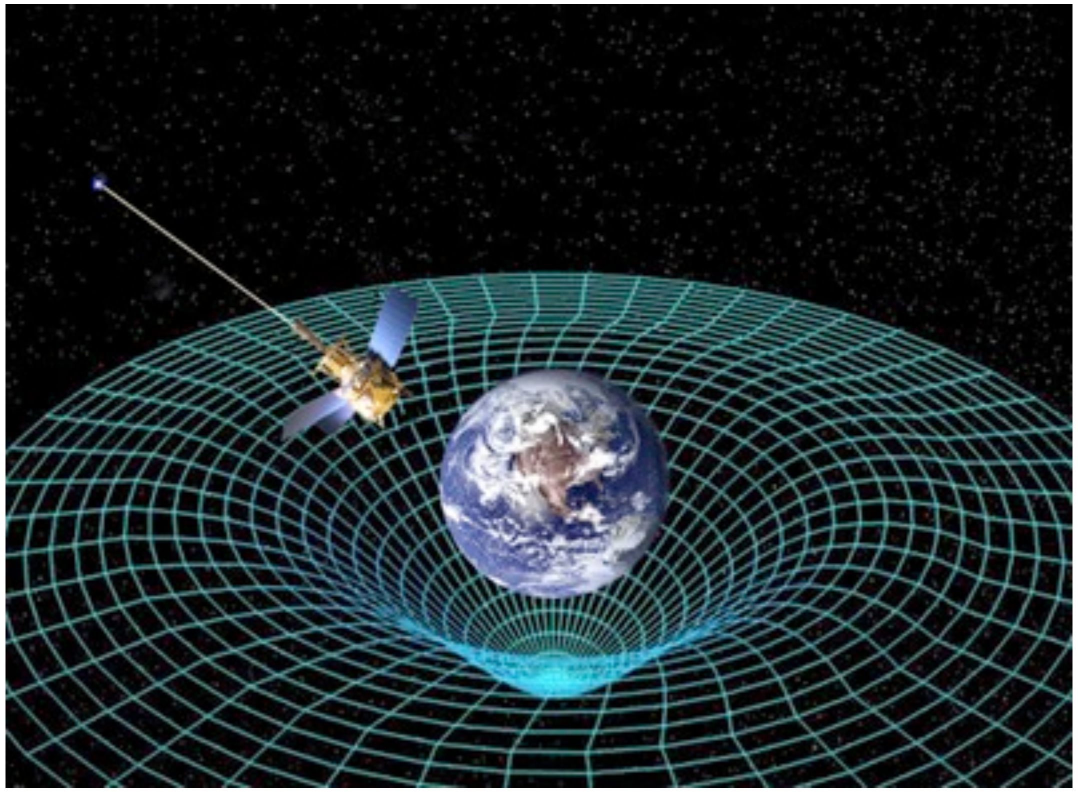
Poincaré model - “space” is the interior of a circle. “Straight lines” are orthogonal to the edges. “Angles” are what you expect.




50°
90°
90°
40°
90°
50°

Our actual spacetime? It varies by location. Globally, we aren’t really sure.





PROPOSITIONAL LOGIC (REALLY CAREFULLY)

Friday, 19 November

SEMANTICS OF PL

- A truth-value assignment is a function (one TVA maps A_1 to True, B to False, C to True, ...)
- Truth-functions are functions (the '$\wedge$' function maps TT to T, TF to F, FT to F, and FF to F, etc.)
- All definitions of important concepts are like this "at least one TVA where all sentences are true" (consistency) and "every TVA where all the premises are true, the conclusion is also true" (entailment)

Sure sounds like FOL!

MODELING PL

- To do this in the formal way done today, we need set theory (Chap 15) but we can cheat a bit and give sound but incomplete axioms

- Also, we can use predicates instead of functions if we wish [like using Mother(a,b) instead of a=mother(b)]

SOME INTENDED INTERPRETATIONS

True(x,y) - x is true on the truth-table row y

False(x,y) - x is false on the truth-table row y

Neg(x,y) - x is the negation of y

Conj(x,y,z) - x is the conjunction of y and z

Disj(x,y,z) - x is the disjunction of y and z

Arrow(x,y,z) - x is a conditional with y as the antecedent and z as the consequent

Darrow(x,y,z) - x is a conditional with y as the left side and z as the right side

SOME AXIOMS

$$\forall x \forall y(\mathrm{True}(x,y) \leftrightarrow \neg \mathrm{False}(x,y))$$

$$\forall x \forall y(\mathrm{Neg}(x,y) \leftrightarrow \forall z(\mathrm{True}(x,z) \leftrightarrow \mathrm{False}(x,z))$$

$$\forall x \forall y \forall z[\mathrm{Conj}(x,y,z) \leftrightarrow \forall w(\mathrm{True}(x,w) \leftrightarrow (\mathrm{True}(y,w) \wedge \mathrm{True}(z,w)))]$$

$$\forall x \forall y \forall z[\mathrm{Disj}(x,y,z) \leftrightarrow \forall w(\mathrm{True}(x,w) \leftrightarrow (\mathrm{True}(y,w) \vee \mathrm{True}(z,w)))]$$

$$\forall x \forall y \forall z[\mathrm{Arrow}(x,y,z) \leftrightarrow \forall w(\mathrm{True}(x,w) \leftrightarrow (\mathrm{True}(y,w) \rightarrow \mathrm{True}(z,w)))]$$

MORE INTENDED INTERPRETATIONS

Taut(x) - x is a tautology

Contra(x) - x is a contradiction

Contin(x) - x is contingent

ProveFrom(x,y) - x is provable from y

Entails(x,y) - x entails y

NoPremises(x) - x provable from no premises

MORE AXIOMS

$\forall x(\mathrm{Taut}(x) \leftrightarrow \forall y\,\mathrm{True}(x,y))$

$\forall x(\mathrm{Contra}(x) \leftrightarrow \forall y\,\mathrm{False}(x,y))$

$\forall x[\mathrm{Contin}(x) \leftrightarrow (\exists y\,\mathrm{False}(x,y) \wedge \exists y\,\mathrm{True}\,(x,y))]$

$\forall x \forall y(\mathrm{Entails}(x,y) \leftrightarrow \mathrm{ProveFrom}(y,x))$

$\forall x \forall y[\mathrm{Entails}(x,y) \leftrightarrow \forall z(\mathrm{True}(x,z) \rightarrow \mathrm{True}(y,z)))]$

$\forall x(\mathrm{NoPremises}(x) \leftrightarrow \forall y\,\mathrm{ProveFrom}(x,y))$

AXIOMS ARE PRETTY STRONG

From these axioms you can prove quite a large number of things

$\forall x \forall y \forall z[(\mathrm{Entails}(x,y) \wedge \mathrm{Entails}(y,z)) \rightarrow \mathrm{Entails}(x,z)]$

Entailment is transitive

$\neg \exists x[\mathrm{Taut}(x) \wedge \exists y(\mathrm{Contin}(y) \wedge \mathrm{Entails}(x,y))]$

No tautology entails any contingent statement

1. $\boxed{a}$ $\boxed{b}$ $\boxed{c}$

2. Entails(a,b) ∧ Entails(b,c)

3. ∀x∀y[Entails(x,y) ↔ ∀z(True(x,z) → True(y,z)))] (AX)

4. Entails(a,b) ↔ ∀z(True(a,z) → True(b,z)) ∀ Elim 3 x2

5. Entails(b,c) ↔ ∀z(True(b,z) → True(c,z)) ∀ Elim 3 x2

6. ∀z(True(a,z) → True(b,z)) Taut Con 2,4

7. ∀z(True(b,z) → True(c,z)) Taut Con 2,5

∀z(True(a,z) → True(c,z)

Entails(a,c) ↔ ∀z(True(a,z) → True(c,z)) ∀ Elim 3 x2

Entails(a,c) ↔ Elim

(Entails(a,b) ∧ Entails(b,c)) → Entails(a,c) → intro

∀x∀y∀z[(Entails(x,y) ∧ Entails(y,z)) → Entails(x,z)] ∀ intro

6. ∀z(True(a,z) → True(b,z)) Taut Con 2,4

7. ∀z(True(b,z) → True(c,z)) Taut Con 2,5

8. $\boxed{t}$

9. True(a,t)

10. True(a,t) → True(b,t) ∀ Elim 6

11. True(b,t) → True(c,t) ∀ Elim 6

True(c,t)

True(a,t) → True(c,t) → intro

∀z(True(a,z) → True(c,z)) ∀ intro

6. ∀z(True(a,z) → True(b,z)) Taut Con 2,4

7. ∀z(True(b,z) → True(c,z)) Taut Con 2,5

8. $\boxed{t}$

9. True(a,t)

10. True(a,t) → True(b,t) ∀ Elim 6

11. True(b,t) → True(c,t) ∀ Elim 6

12. True(c,t) Taut Con 9,10,11

13. True(a,t) → True(c,t) → intro 9-12

14. ∀z(True(a,z) → True(c,z) ∀ intro 8-13

AXIOMS ARE PRETTY STRONG

From these axioms you can prove quite a large number of things -- These are more annoying

$\forall x[\mathrm{Taut}(x) \rightarrow \mathrm{NoPremises}(x)]$

Tautologies are provable from no premises

$\forall x[\mathrm{Contra}(x) \rightarrow \forall y\ \mathrm{ProveFrom}(y,x)]$

Any formula is provable from a contradiction

INFORMAL PROOF

Proof that any formula follows from a contradiction

First, note that a contradiction entails any formula whatever. This is because for an arbitrary formula P, it is vacuously true that any TVA which makes the contradiction true (there are none) will also make P true. By the completeness theorem, since the contradiction entails P, P is provable from it. Since P was arbitrary, any formula whatever is provable from a contradiction.

1. $\boxed{a}$

2. $\mathrm{Contra}(a)$

3. $\boxed{b}$

4. $\forall x(\mathrm{Contra}(x) \leftrightarrow \forall y\ \mathrm{False}(x,y))$ AXIOM

5. $\mathrm{Contra}(a) \leftrightarrow \forall y\ \mathrm{False}(a,y))$ $\forall$ Elim 4

6. $\forall y\ \mathrm{False}(a,y)$ $\leftrightarrow$ Elim 2,5

$\mathrm{Entails}(a,b)$ -- Need

$\forall x \forall y(\mathrm{Entails}(x,y) \leftrightarrow \mathrm{ProveFrom}(y,x))$ AXIOM

$\mathrm{Entails}(a,b) \leftrightarrow \mathrm{ProveFrom}(b,a)$ $\forall$ Elim x2

$\mathrm{ProveFrom}(b,a)$

$\forall y\ \mathrm{ProveFrom}(y,a)$ $\forall$ intro

$\mathrm{Contra}(a) \rightarrow \forall y\ \mathrm{ProveFrom}(y,a)$ $\rightarrow$ intro

$\forall x[\mathrm{Contra}(x) \rightarrow \forall y\ \mathrm{ProveFrom}(y,x)]$ $\forall$ intro

6. $\forall y\,\mathrm{False}(a,y)$ ↔ Elim 2,5

7. $\boxed{c}$

8. $\mathrm{False}(a,c)$ ∀ Elim 6

9. $\forall x \forall y(\mathrm{True}(x,y) \leftrightarrow \neg\mathrm{False}(x,y))$ AXIOM

10. $\mathrm{True}(a,c) \leftrightarrow \neg\mathrm{False}(a,c)$ ∀ Elim 9 x2

$\mathrm{True}(a,c) \rightarrow \mathrm{True}(b,c)$

$\forall z(\mathrm{True}(a,z) \rightarrow \mathrm{True}(b,z))$ ∀ intro

$\forall x \forall y[\mathrm{Entails}(x,y) \leftrightarrow \forall z(\mathrm{True}(x,z) \rightarrow \mathrm{True}(y,z))]$ (AX)

$\mathrm{Entails}(a,b) \leftrightarrow \forall z(\mathrm{True}(a,z) \rightarrow \mathrm{True}(b,z))$ ∀ Elim

$\mathrm{Entails}(a,b)$ -- Need

6. $\forall y\,\mathrm{False}(a,y)$ ↔ Elim 2,5

7. $\boxed{c}$

8. $\mathrm{False}(a,c)$ ∀ Elim 6

9. $\forall x \forall y(\mathrm{True}(x,y) \leftrightarrow \neg \mathrm{False}(x,y))$ AXIOM

10. $\mathrm{True}(a,c) \leftrightarrow \neg \mathrm{False}(a,c)$ ∀ Elim 9 x2

11. $\mathrm{True}(a,c) \rightarrow \mathrm{True}(b,c)$ Taut Con 8,10

12. $\forall z(\mathrm{True}(a,z) \rightarrow \mathrm{True}(b,z))$ ∀ intro 7-11

13. $\forall x \forall y[\mathrm{Entails}(x,y) \leftrightarrow \forall z(\mathrm{True}(x,z) \rightarrow \mathrm{True}(y,z))]$ (AX)

14. $\mathrm{Entails}(a,b) \leftrightarrow \forall z(\mathrm{True}(a,z) \rightarrow \mathrm{True}(b,z))$ ∀ Elim 14

15. $\mathrm{Entails}(a,b)$ ↔ Elim 12,14