PUZZLE

Is this answer to this question “no”?

Independence and Logical Strength

Monday, 15 November

INDEPENDENCE

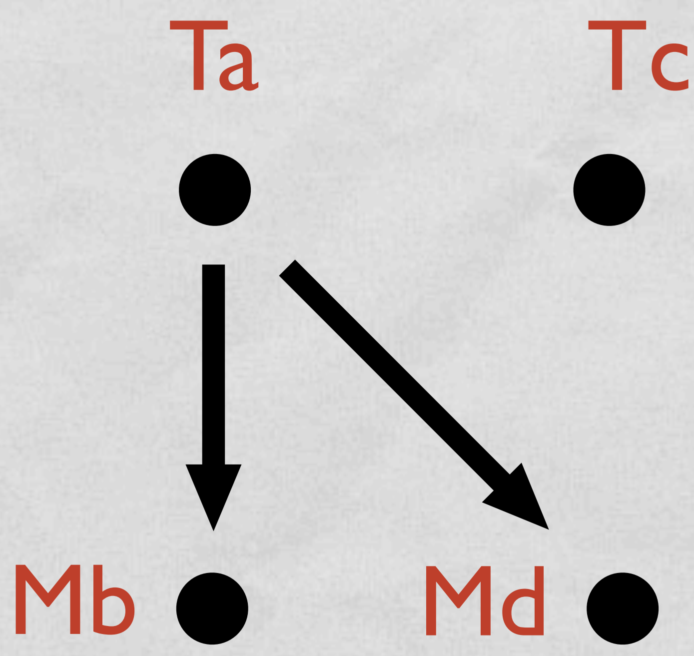
- If a set of premises {P1...Pn} $\nvdash$ A and $\nvdash \neg$A then we say that A is <u>independent</u> of {P1...Pn}.
- A is independent of {P1...Pn} if and only if {P1...Pn, A} and {P1...Pn, ¬A} are both consistent.
- To show that a sentence is independent of some premises, we need two interpretations. Both make the premises true and one makes the conclusion true and one makes it false.

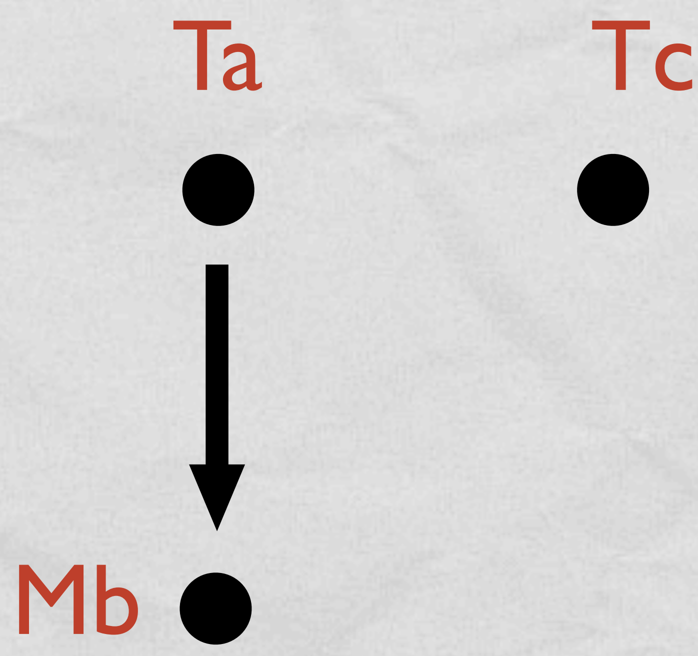
1. $\exists x(T(x) \wedge \forall y(M(y) \rightarrow A(x,y))$

2. $\exists x(T(x) \wedge \forall y(M(y) \rightarrow \neg A(x,y))$

3. $\exists x(T(x) \wedge \exists y \exists z(y \neq z \wedge M(y) \wedge M(z) \wedge A(x,y) \wedge A(x,z))$

Show that 3 is independent of 1+2





T, T, T

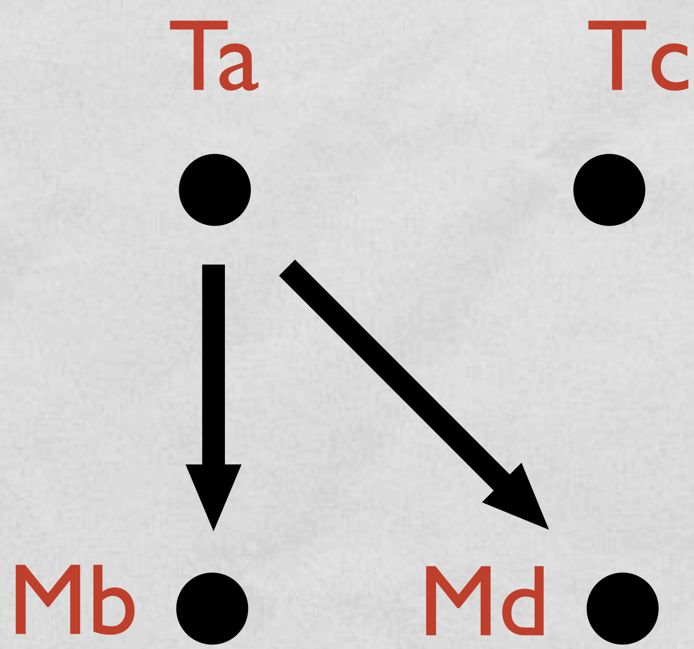
T, T, F

1. $\exists x(T(x) \wedge \forall y(M(y) \rightarrow A(x,y))$

2. $\exists x(T(x) \wedge \forall y(M(y) \rightarrow \neg A(x,y))$

3. $\exists x(T(x) \wedge \exists y \exists z(y \neq z \wedge M(y) \wedge M(z) \wedge A(x,y) \wedge A(x,z))$

Show that 3 is independent of 1+2



T, T, T

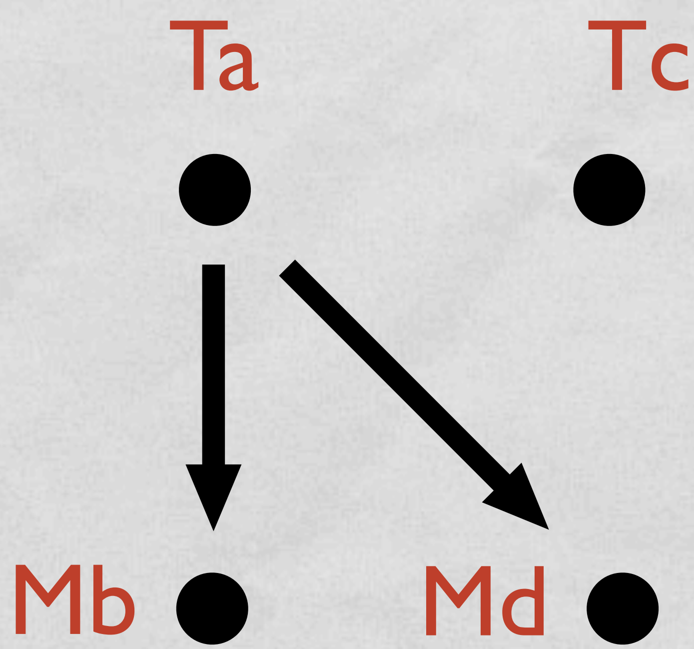
T, T, F

1. $\exists x(T(x) \wedge \forall y(M(y) \rightarrow A(x,y))$

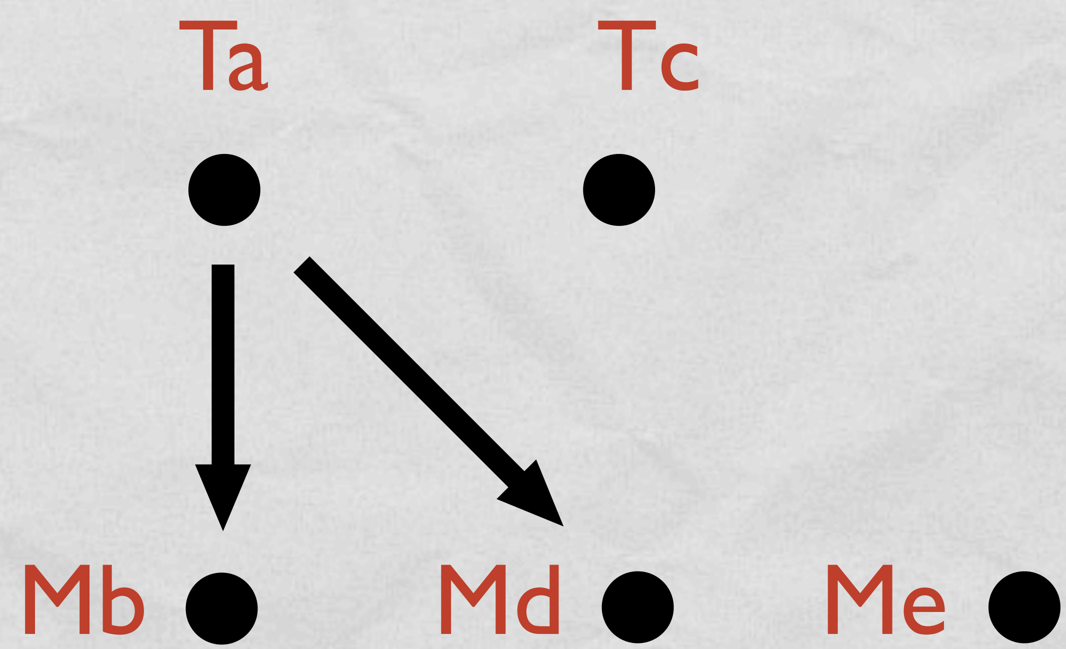
2. $\exists x(T(x) \wedge \forall y(M(y) \rightarrow \neg A(x,y))$

3. $\exists x(T(x) \wedge \exists y \exists z(y \neq z \wedge M(y) \wedge M(z) \wedge A(x,y) \wedge A(x,z))$

Show that 1 is independent of 2+3



T, T, T



~~T~~, T, T

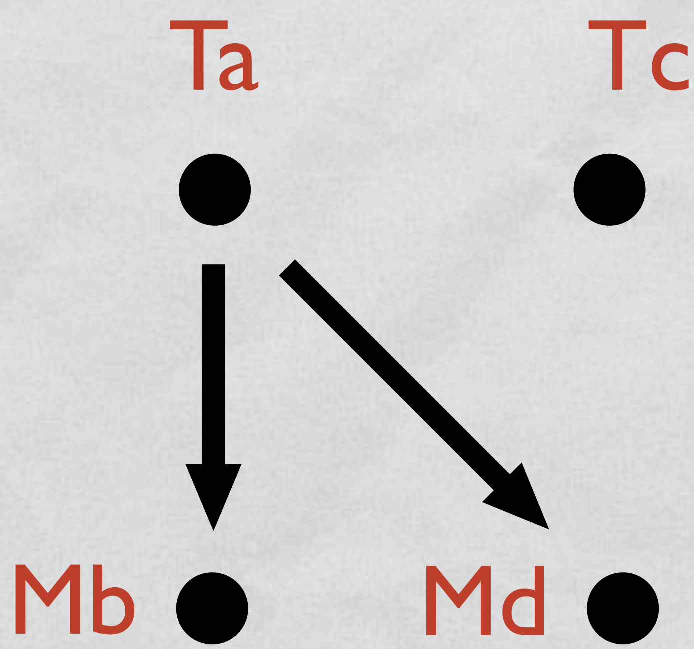
F, T, T

1. $\exists x(T(x) \wedge \forall y(M(y) \rightarrow A(x,y))$

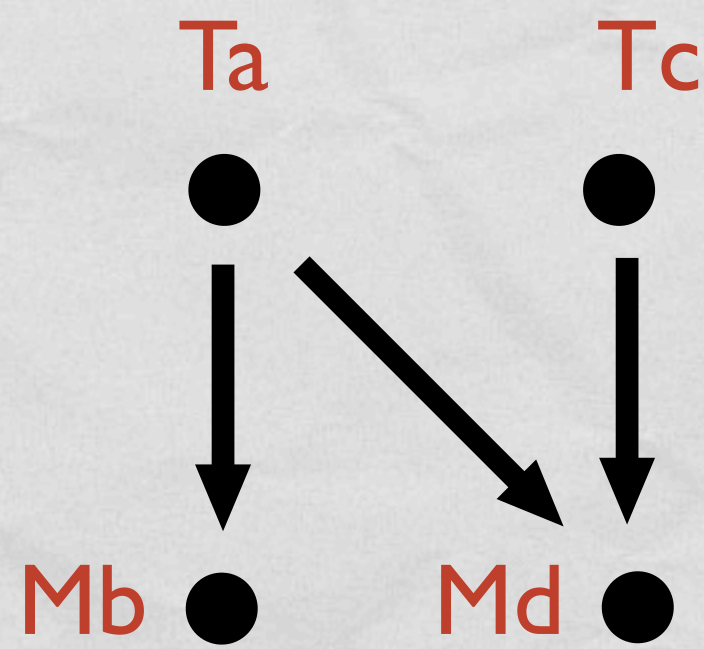
2. $\exists x(T(x) \wedge \forall y(M(y) \rightarrow \neg A(x,y))$

3. $\exists x(T(x) \wedge \exists y \exists z(y \neq z \wedge M(y) \wedge M(z) \wedge A(x,y) \wedge A(x,z))$

Show that 2 is independent of 1+3



T, T, T



T, ~~T~~, T

T, F, T

Mutual Independence

- A set of sentences is mutually independent if each sentence is independent of the others.
- To show that {P1, P2, P3} are mutually independent requires four interpretations - TTT, TTF, TFT, FTT
- To show that n sentences are mutually independent requires n+1 interpretations - show that the whole set is consistent and that each could be false while the others are still true.

1. $\exists x \exists y(T(x) \land T(y) \land \forall z(M(z) \rightarrow (A(x,z) \land A(y,z))))$
2. $\exists x \exists y(T(x) \land T(y) \land \forall z(M(z) \rightarrow (A(x,z) \leftrightarrow A(y,z))))$
3. $\exists x \exists y(T(x) \land T(y) \land \forall z(M(z) \rightarrow (A(x,z) \rightarrow A(y,z))))$
4. $\exists x \exists y(T(x) \land T(y) \land \forall z(M(z) \rightarrow (A(x,z) \lor A(y,z))))$

These are obviously not mutually independent

Logical Strength

- A sentence P is logically stronger than Q iff $P \vdash Q$ but $Q \nvdash P$.
- P is weaker than Q iff Q is stronger than P.
- For any two sentences there are only four possibilities: Either P is stronger than Q, weaker than Q, equivalent to Q, or P and Q are mutually independent.

1. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \wedge A(y,z))))$
2. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \leftrightarrow A(y,z))))$
3. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \rightarrow A(y,z))))$
4. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \vee A(y,z))))$

These are not mutually independent

1 is stronger than 2 is stronger than 3

'Stronger than' is transitive:

$$\forall x \forall y \forall z((S(x,y) \wedge S(y,z)) \rightarrow S(x,z))$$

1 is stronger than 4

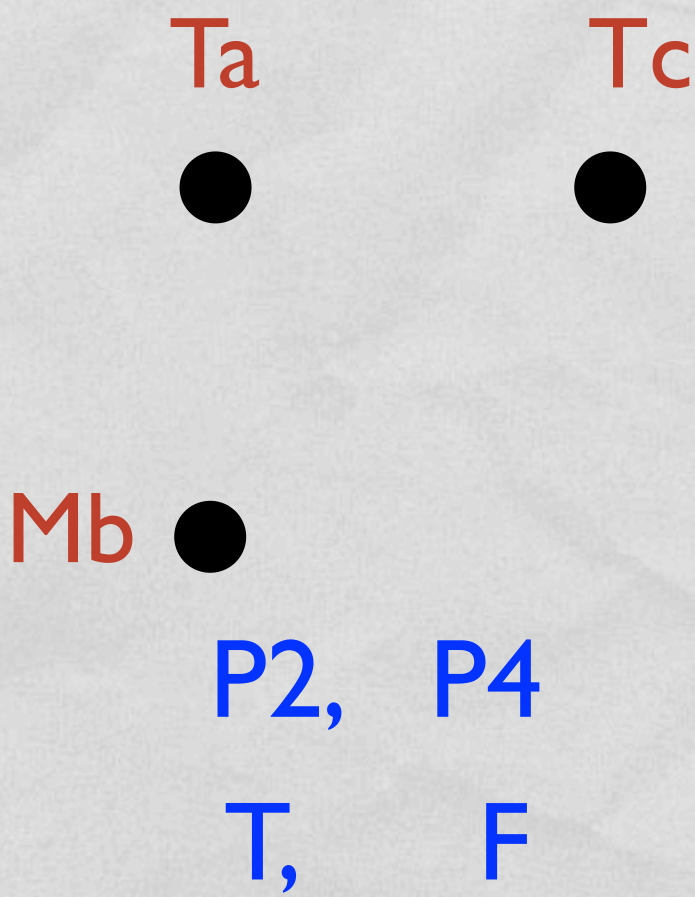
1. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \wedge A(y,z))))$

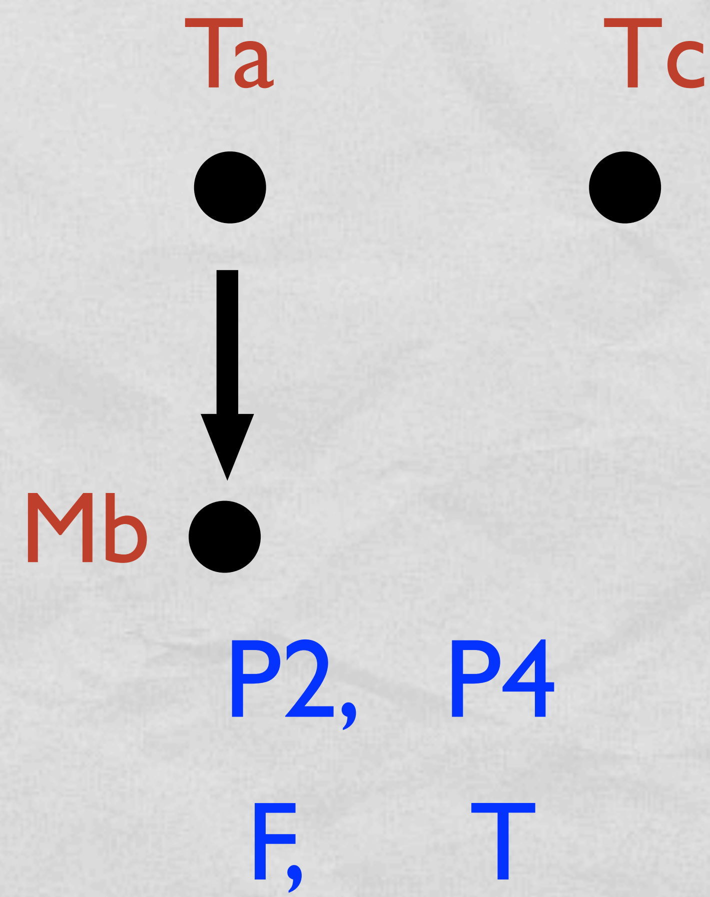
2. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \leftrightarrow A(y,z))))$

3. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \rightarrow A(y,z))))$

4. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \vee A(y,z))))$

What about 2, 4?





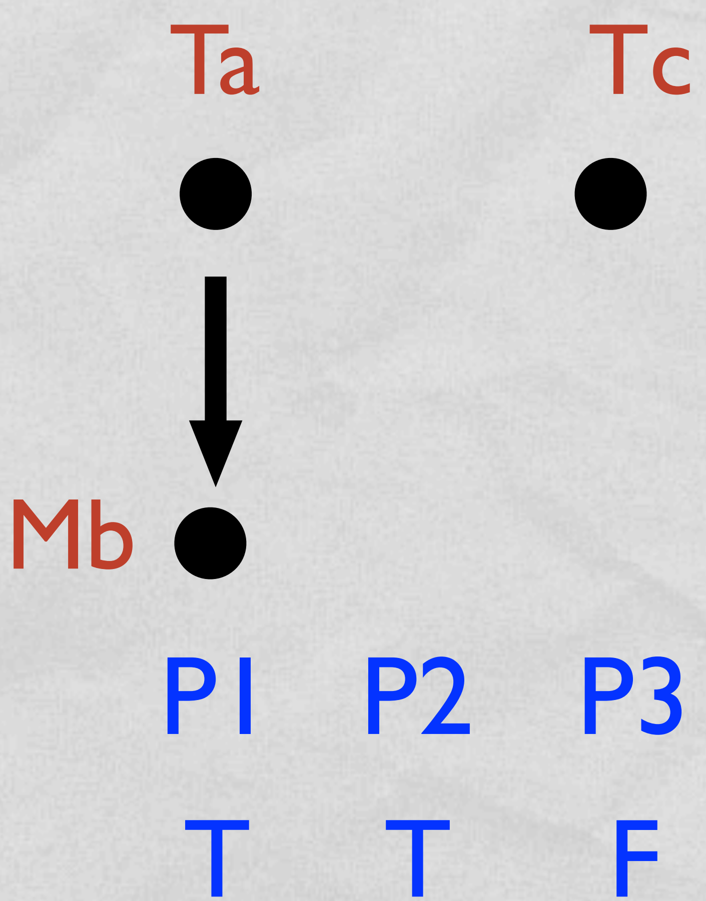
1. $\exists x \exists y(T(x) \wedge T(y) \wedge \exists z(M(z) \wedge A(x,z) \wedge \neg A(y,z)))$

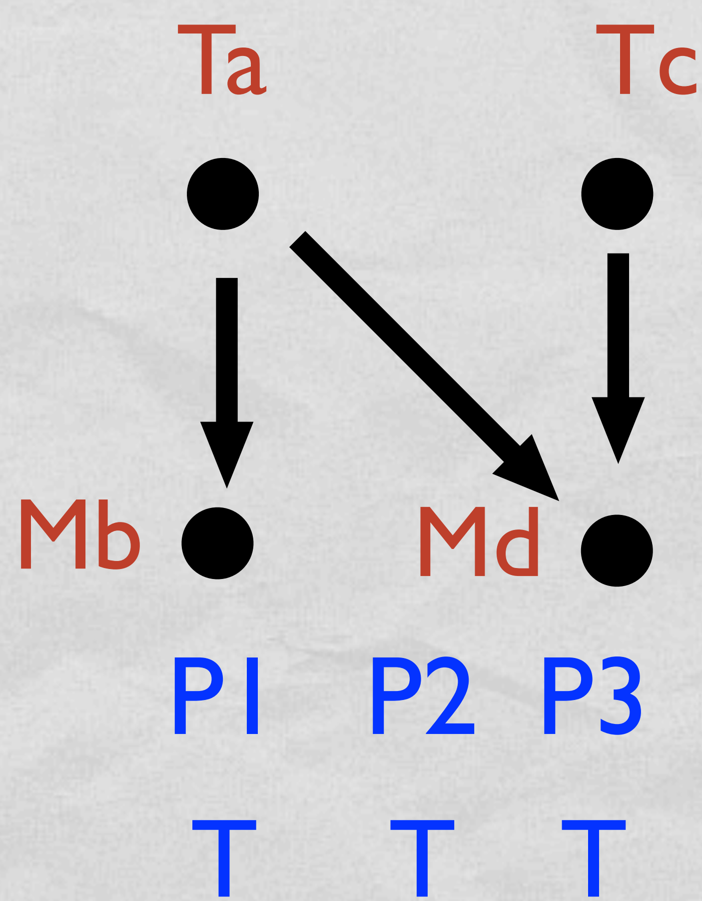
2. $\exists x(M(x) \wedge T(a) \wedge A(a,x) \wedge \forall z((T(z) \wedge a \neq z) \rightarrow \neg A(x,z)))$

3. $\forall x \forall y((T(x) \wedge T(y) \wedge x \neq y) \rightarrow \exists z(M(z) \wedge A(x,z) \wedge A(y,z)))$

¬3. $\exists x \exists y(T(x) \wedge T(y) \wedge x \neq y \wedge \forall z(M(z) \rightarrow (\neg A(x,z) \vee \neg A(y,z))))$

Independent?





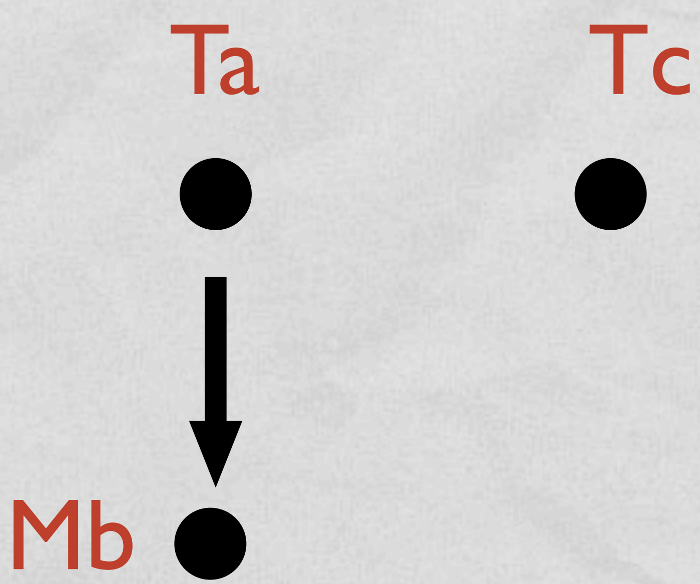
1. $\exists x \exists y(T(x) \wedge T(y) \wedge \exists z(M(z) \wedge A(x,z) \wedge \neg A(y,z)))$

2. $\exists x(M(x) \wedge T(a) \wedge A(a,x) \wedge \forall z((T(z) \wedge a \neq z) \rightarrow \neg A(x,z)))$

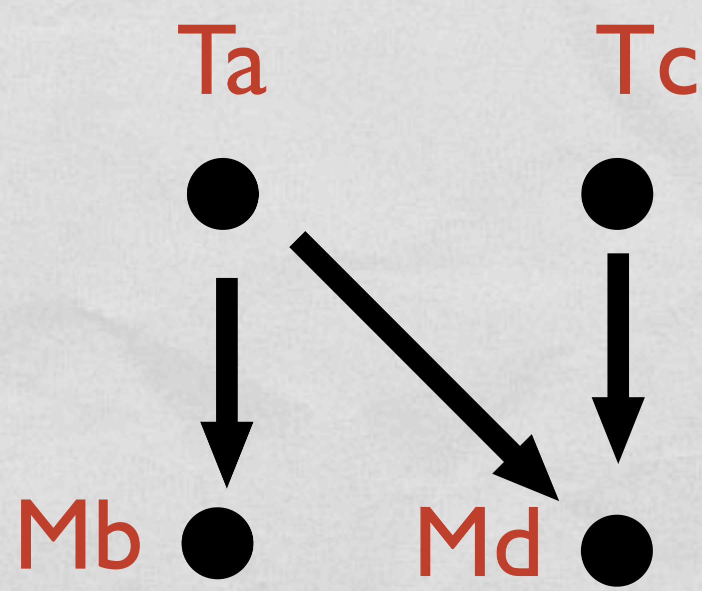
3. $\forall x \forall y((T(x) \wedge T(y) \wedge x \neq y) \rightarrow \exists z(M(z) \wedge A(x,z) \wedge A(y,z)))$

¬3. $\exists x \exists y(T(x) \wedge T(y) \wedge x \neq y \wedge \forall z(M(z) \rightarrow (\neg A(x,z) \vee \neg A(y,z))))$

So P3 is independent of 1+2



P1	P2	P3
T	T	F



P1	P2	P3
T	T	T

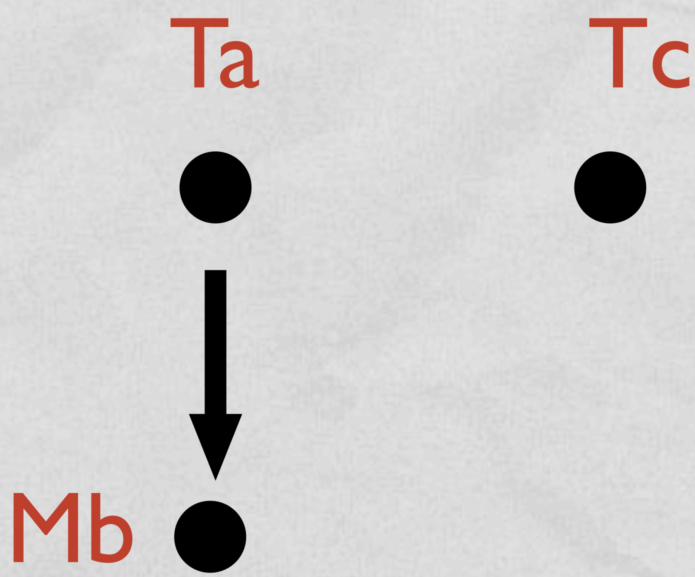
1. $\exists x \exists y (T(x) \wedge T(y) \wedge \exists z (M(z) \wedge A(x,z) \wedge \neg A(y,z)))$

2. $\exists x (M(x) \wedge T(a) \wedge A(a,x) \wedge \forall z ((T(z) \wedge a \neq z) \rightarrow \neg A(x,z)))$

¬2. $\forall x ((M(x) \wedge T(a) \wedge A(a,x)) \rightarrow \exists z (T(z) \wedge a \neq z \wedge A(x,z)))$

3. $\forall x \forall y ((T(x) \wedge T(y) \wedge x \neq y) \rightarrow \exists z (M(z) \wedge A(x,z) \wedge A(y,z)))$

Is P2 independent of 1+3?



P1	P2
T	T

1. $\exists x \exists y(T(x) \wedge T(y) \wedge \exists z(M(z) \wedge A(x,z) \wedge \neg A(y,z)))$

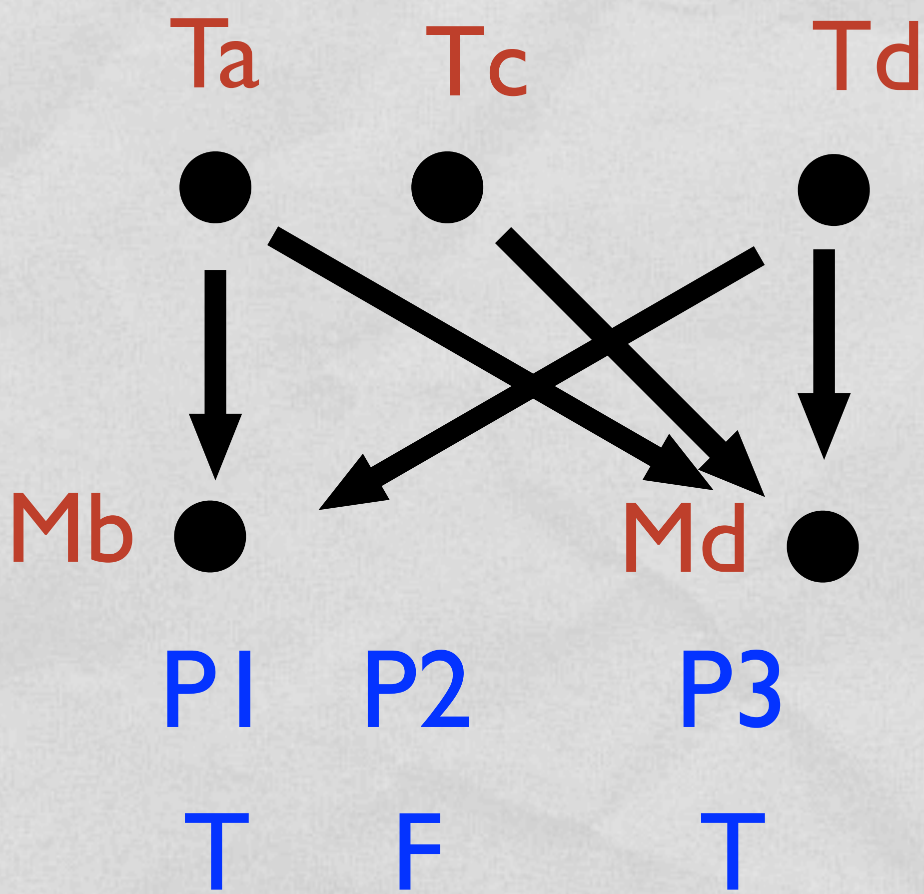
2. $\exists x(M(x) \wedge T(a) \wedge A(a,x) \wedge \forall z((T(z) \wedge a \neq z) \rightarrow \neg A(x,z)))$

¬2. $\forall x((M(x) \wedge T(a) \wedge A(a,x)) \rightarrow \exists z(T(z) \wedge a \neq z \wedge A(x,z)))$

3. $\forall x \forall y((T(x) \wedge T(y) \wedge x \neq y) \rightarrow \exists z(M(z) \wedge A(x,z) \wedge A(y,z)))$

Is P2 independent of 1+3?

Yes



1. $\exists x \exists y(T(x) \wedge T(y) \wedge \exists z(M(z) \wedge A(x,z) \wedge \neg A(y,z)))$

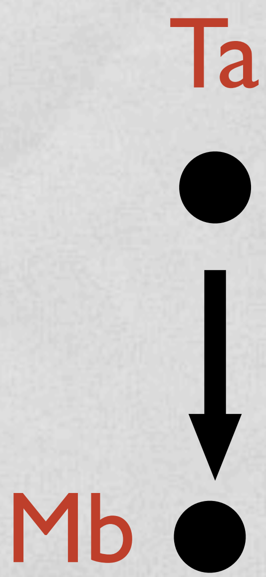
¬1. $\forall x \forall y((T(x) \wedge T(y)) \rightarrow \forall z(M(z) \rightarrow (A(x,z) \leftrightarrow A(y,z))))$

2. $\exists x(M(x) \wedge T(a) \wedge A(a,x) \wedge \forall z((T(z) \wedge a \neq z) \rightarrow \neg A(x,z)))$

3. $\forall x \forall y((T(x) \wedge T(y) \wedge x \neq y) \rightarrow \exists z(M(z) \wedge A(x,z) \wedge A(y,z)))$

Is P1 independent of 2+3?

Yes



P1	P2	P3
F	T	T

Logical Consequence - Revisited

- Recall that an argument is logically valid iff the conclusion is a logical consequence of the premises.
- This comes apart from FO consequence when there is some crucial facts about the meaning of the predicates in the sentences.
- Example: Every cube is to the right of any dodec. Therefore, Every dodec is to the left of any cube. This is not FO valid.

Logical Consequence - Revisited

- However, you can often turn a valid argument into an FO-valid one by adding some explicit premise about how the predicate matters.
- For example, adding the claim that If x is to the right of y, then y is to the left of x in the previous argument.

$\forall x \forall y((Cube(x) \land Dodec(y)) \to RightOf(x,y))$

$\nvdash \forall x \forall y((Cube(x) \land Dodec(y)) \to LeftOf(y,x))$

Because

$\forall x \forall y((C(x) \land D(y)) \to R(x,y))$

$\nvdash \forall x \forall y((C(x) \land D(y)) \to L(y,x))$

FO consequence just pays attention to the connectives and quantifiers (and identity)

However

$\forall x \forall y(RightOf(x,y) \to LeftOf(y,x))$

$\forall x \forall y((Cube(x) \land Dodec(y)) \to RightOf(x,y))$

$\vdash \forall x \forall y((Cube(x) \land Dodec(y)) \to LeftOf(y,x))$

The Axiomatic Method

- Sentences that reflect the meaning of predicates we want to take into account are called meaning postulates.

- These are a kind of axiom: a claim accepted as true for some domain, which is then used as the basis for arguments to establish other truths of that domain.

- The axiomatic method is the method of defining axioms for a certain domain in order to bridge the gap between (intuitive) logical consequence and (technical) first-order consequence.

The Axiomatic Method

- The Shape Axioms for TW:
  1. $\neg\exists x(Cube(x) \wedge Dodec(x))$
  2. $\neg\exists x(Tet(x) \wedge Dodec(x))$
  3. $\neg\exists x(Cube(x) \wedge Tet(x))$
  4. $\forall x(Cube(x) \vee Dodec(x) \vee Tet(x))$
- First three axioms come from the meaning of shape. Nothing can be two different shapes (simultaneously).
- Fourth axiom is not part of the meaning of shape, but it is true of how shape works in Tarski's World.

The Axiomatic Method

- With the shape axioms as premises, we can turn more cases of logical consequence into first-order consequences.

- $\neg\exists x\ Cube(x)$ therefore $\forall x(Dodec(x) \leftrightarrow \neg Tet(x))$

- $\neg\exists x\ C(x)$ (Premise)

  $\forall x(C(x) \lor D(x) \lor T(x))$ (Axiom 4)

  $\neg\exists x(T(x) \land D(x))$ (Axiom 2)

  $\forall x(D(x) \leftrightarrow \neg T(x))$

Logic, Meaning and Worlds

- When we reason, we have background assumptions.
- Tautological relationships reflect the (fixed) meanings of truth-functional connectives.
- First-order relationships also reflect the (fixed) meanings of identity and quantifiers.
- Logical/analytic relationships also take as fixed and reflect what we mean by predicates (meaning postulates).
- Wider relationships still can take into account features of particular domains (other axioms).