The Hardest Logic Puzzle Ever

Three gods A, B, and C are called, in some order, True, False, and Random. True always speaks truly, False always speaks falsely, but whether Random speaks truly or falsely is a completely random matter. Your task is to determine the identities of A, B, and C by asking three yes/no questions; each question must be put to exactly one god. The gods understand English, but will answer all questions in their own language, in which the words for *yes* and *no* are 'da' and 'ja', in some order. You do not know which word means which.

AVOID THE RANDOM

This is pretty hard. I will give you the question, you figure out what to do.

---Question (asked to god A): Are you a knight iff B is random iff ‘Bal’ means ‘yes’?

What if they answer Bal?
What if they answer Da?

Avoid the Random

If A is a knight then she says ‘Bal’ iff Knight(a) iff B is random so iff B is random.

If A is a knave then she says ‘Da’ iff Knight(a) iff B is random so ‘Bal’ iff B is random.

Either way, if they say ‘Bal’ then A/C are knight/knave, ‘Da’ then A/B are knight/knave.

If A is random, then B/C are knight knave no matter what.

So if you hear “Bal”, C is a knight/knave.
If you hear “Da”, B is a knight/knave.

Putting it Together

First question (asked to god A): Are you a knight iff B is random iff 'Bal' means 'yes'?

- If they say "Bal", go to C
- If they say "Da", go to B

Second question (asked to B or C from above): Does 'Bal' mean 'yes' iff 2+2=4?

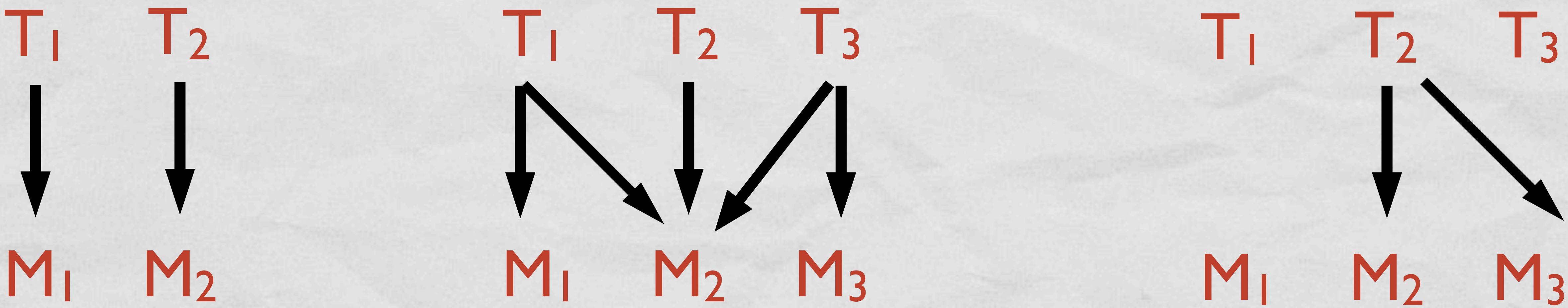
- If they say "Bal", then they are a knight.
- If they say "Da", then they are a knave.

Third question (asked to same god as Q2):

- Does 'Bal' mean 'yes' iff A is Random?

Independence and Logical Strength

Friday, 12 November



$$\exists x \exists y (T(x) \wedge T(y) \wedge \forall z (M(z) \rightarrow (A(x,z) \vee A(y,z))))$$

True, True, False

$$\forall x \forall y ((M(x) \wedge M(y)) \rightarrow \exists z (T(z) \wedge (A(z,x) \leftrightarrow A(z,y))))$$

False, False, True

$$\exists x (T(x) \wedge \forall y ((M(y) \wedge \neg A(x,y)) \rightarrow \forall z \neg A(z,y)))$$

False, False, True

INDEPENDENCE

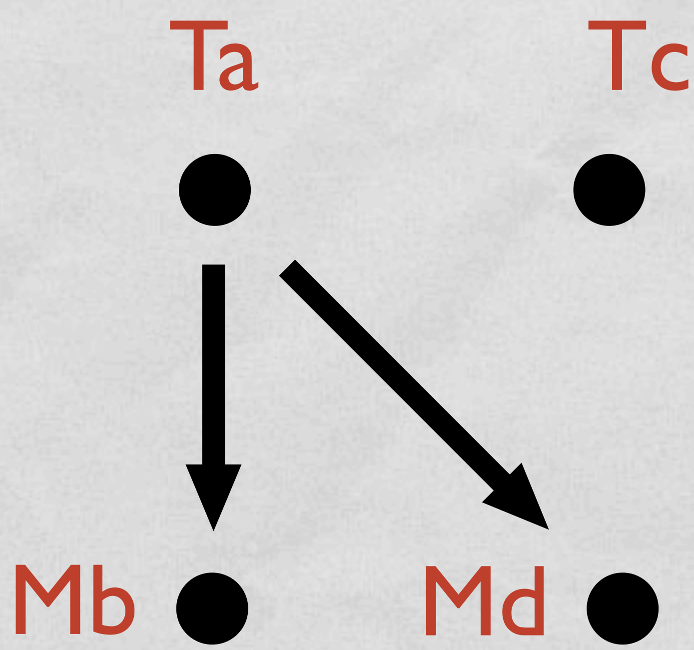
- If a set of premises {P1...Pn} $\nvdash$ A and $\nvdash$ ¬A then we say that A is <u>independent</u> of {P1...Pn}.
- A is independent of {P1...Pn} if and only if {P1...Pn, A} and {P1...Pn, ¬A} are both consistent.
- To show that a sentence is independent of some premises, we need two interpretations. Both make the premises true and one makes the conclusion true and one makes it false.

1. $\exists x(T(x) \wedge \forall y(M(y) \rightarrow A(x,y))$

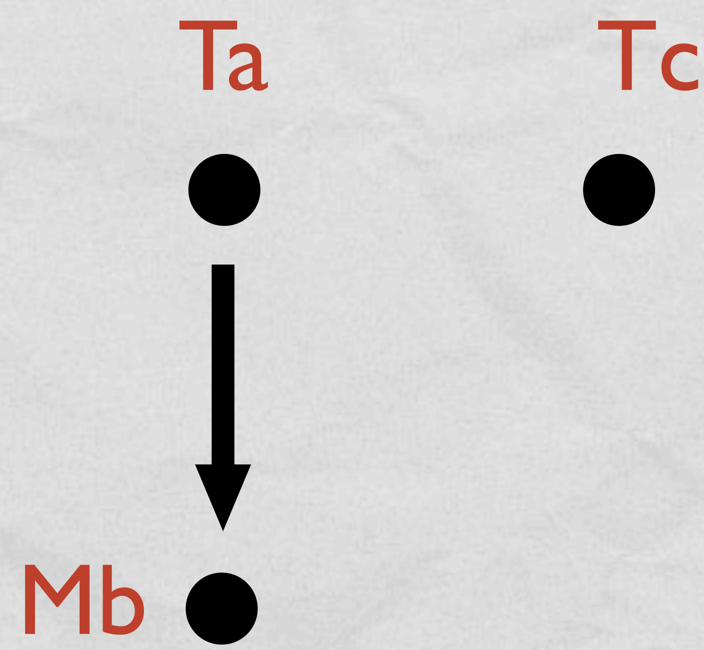
2. $\exists x(T(x) \wedge \forall y(M(y) \rightarrow \neg A(x,y))$

3. $\exists x(T(x) \wedge \exists y \exists z(y \neq z \wedge M(y) \wedge M(z) \wedge A(x,y) \wedge A(x,z))$

Show that 3 is independent of 1+2



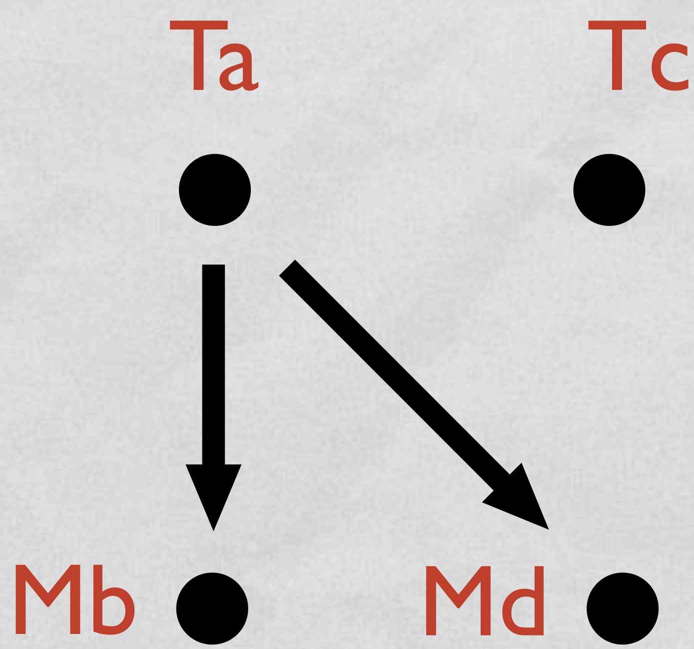
T, T, T



T, T, F

1. $\exists x(T(x) \land \forall y(M(y) \rightarrow A(x,y))$
2. $\exists x(T(x) \land \forall y(M(y) \rightarrow \neg A(x,y))$
3. $\exists x(T(x) \land \exists y \exists z(y \neq z \land M(y) \land M(z) \land A(x,y) \land A(x,z))$

Show that 3 is independent of 1+2



T, T, T

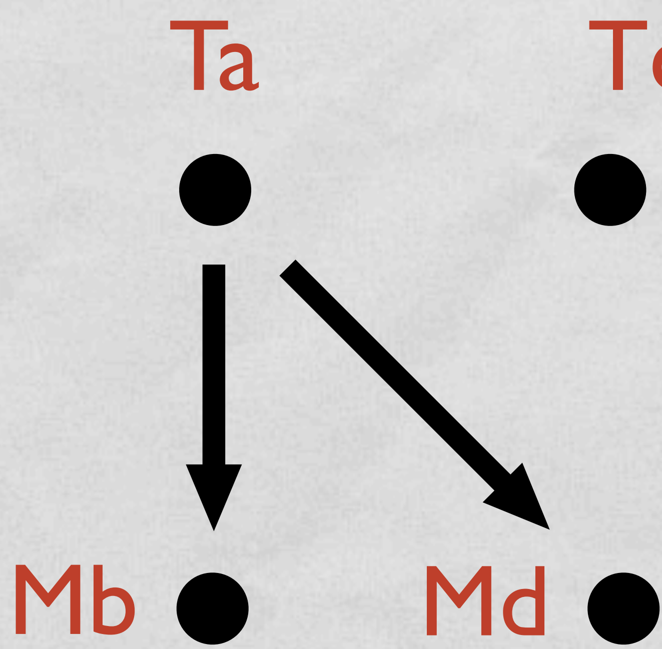
T, T, F

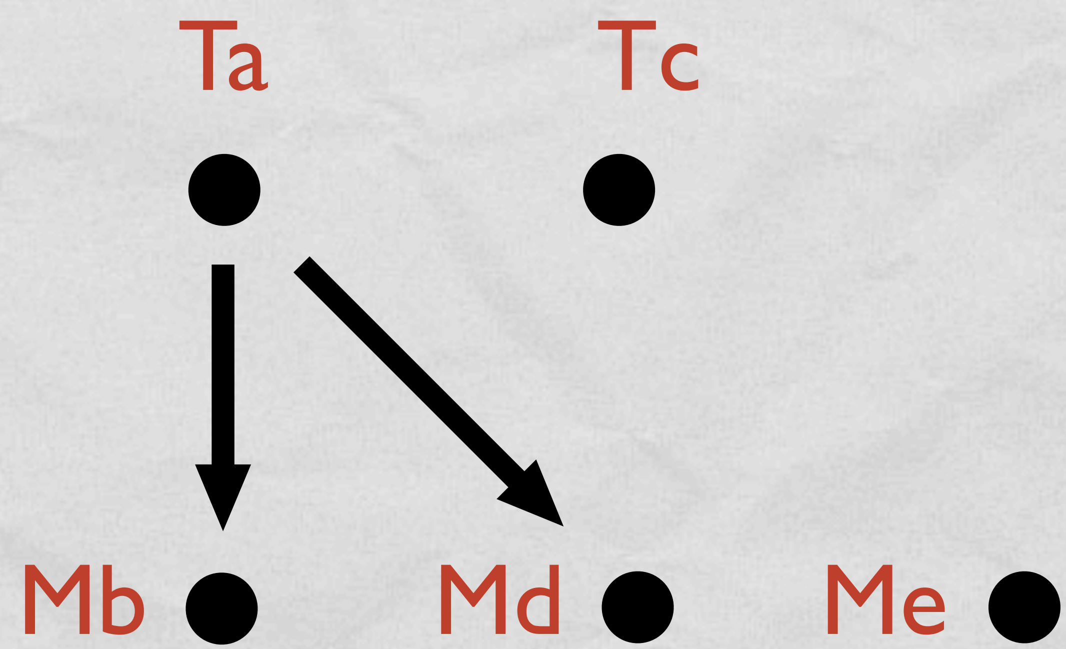
1. $\exists x(T(x) \wedge \forall y(M(y) \rightarrow A(x,y))$

2. $\exists x(T(x) \wedge \forall y(M(y) \rightarrow \neg A(x,y))$

3. $\exists x(T(x) \wedge \exists y \exists z(y \neq z \wedge M(y) \wedge M(z) \wedge A(x,y) \wedge A(x,z))$

Show that 1 is independent of 2+3





T, T, T

~~T~~, T, T

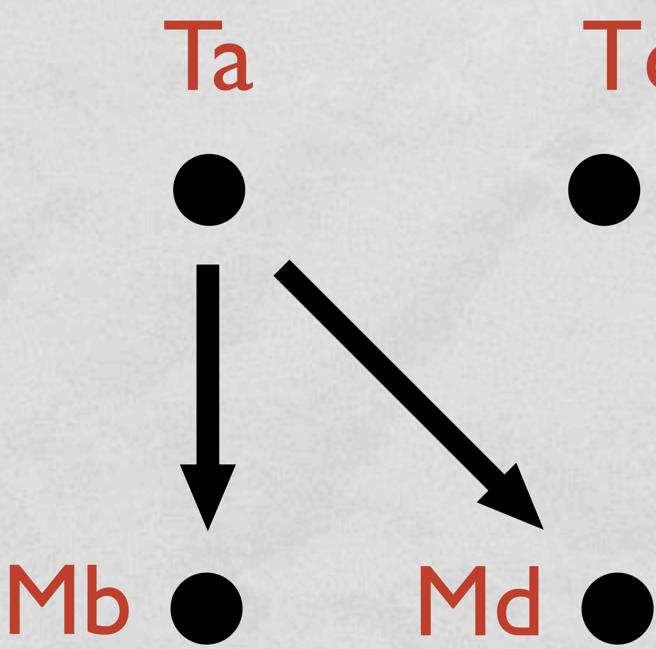
F, T, T

1. $\exists x(T(x) \land \forall y(M(y) \rightarrow A(x,y))$

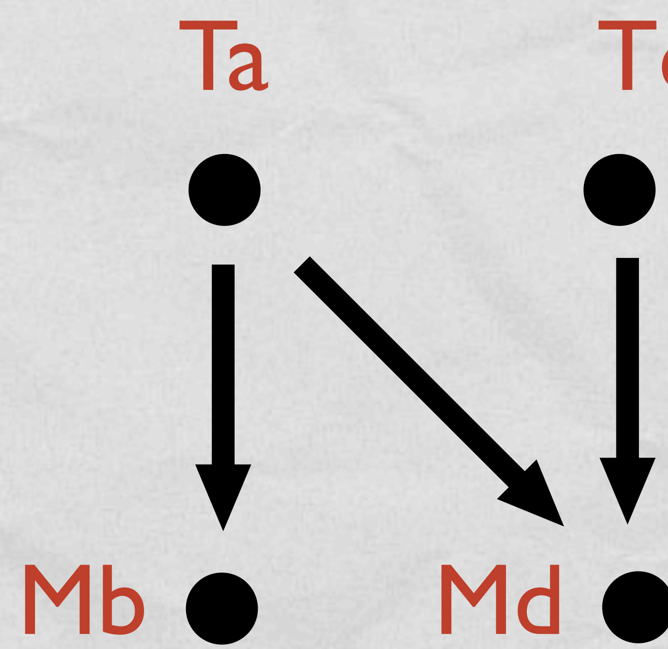
2. $\exists x(T(x) \land \forall y(M(y) \rightarrow \neg A(x,y))$

3. $\exists x(T(x) \land \exists y \exists z(y \neq z \land M(y) \land M(z) \land A(x,y) \land A(x,z))$

Show that 2 is independent of 1+3



T, T, T



T, ~~T~~, T

T, F, T

Mutual Independence

- A set of sentences is mutually independent if each sentence is independent of the others.
- To show that {P1, P2, P3} are mutually independent requires four interpretations - TTT, TTF, TFT, FTT
- To show that n sentences are mutually independent requires n+1 interpretations - show that the whole set is consistent and that each could be false while the others are still true.

1. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \wedge A(y,z))))$
2. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \leftrightarrow A(y,z))))$
3. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \rightarrow A(y,z))))$
4. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \vee A(y,z))))$

These are obviously not mutually independent

Logical Strength

- A sentence P is logically stronger than Q iff $P \vdash Q$ but $Q \nvdash P$.

- P is weaker than Q iff Q is stronger than P.

- For any two sentences there are only four possibilities: Either P is stronger than Q, weaker than Q, equivalent to Q, or P and Q are mutually independent.

1. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \wedge A(y,z))))$
2. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \leftrightarrow A(y,z))))$
3. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \rightarrow A(y,z))))$
4. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \vee A(y,z))))$

These are not mutually independent

1 is stronger than 2 is stronger than 3

'Stronger than' is transitive:

$$\forall x \forall y \forall z((S(x,y) \wedge S(y,z)) \rightarrow S(x,z))$$

1 is stronger than 4

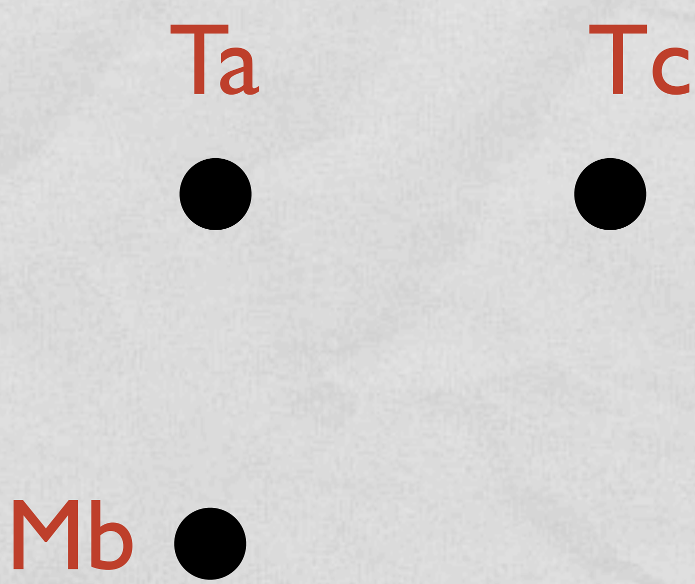
1. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \wedge A(y,z))))$

2. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \leftrightarrow A(y,z))))$

3. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \rightarrow A(y,z))))$

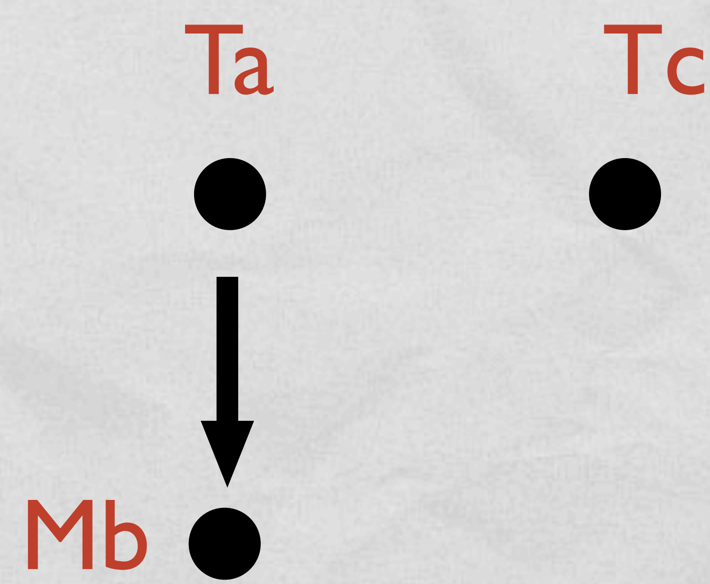
4. $\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \vee A(y,z))))$

What about 2, 4?



P2, P4

T, F



P2, P4

F, T

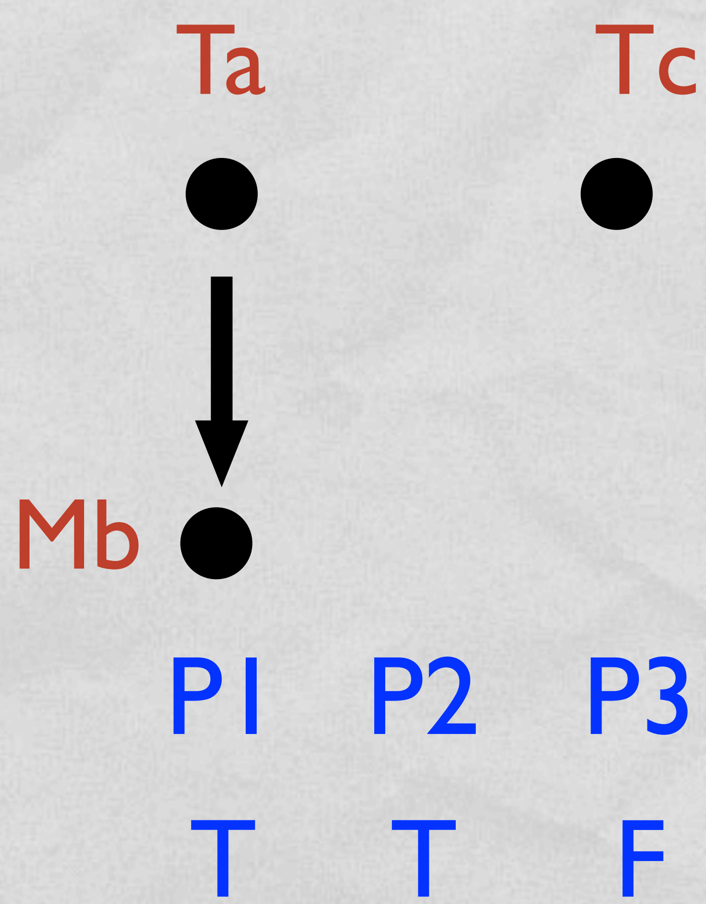
1. $\exists x \exists y(T(x) \land T(y) \land \exists z(M(z) \land A(x,z) \land \neg A(y,z)))$

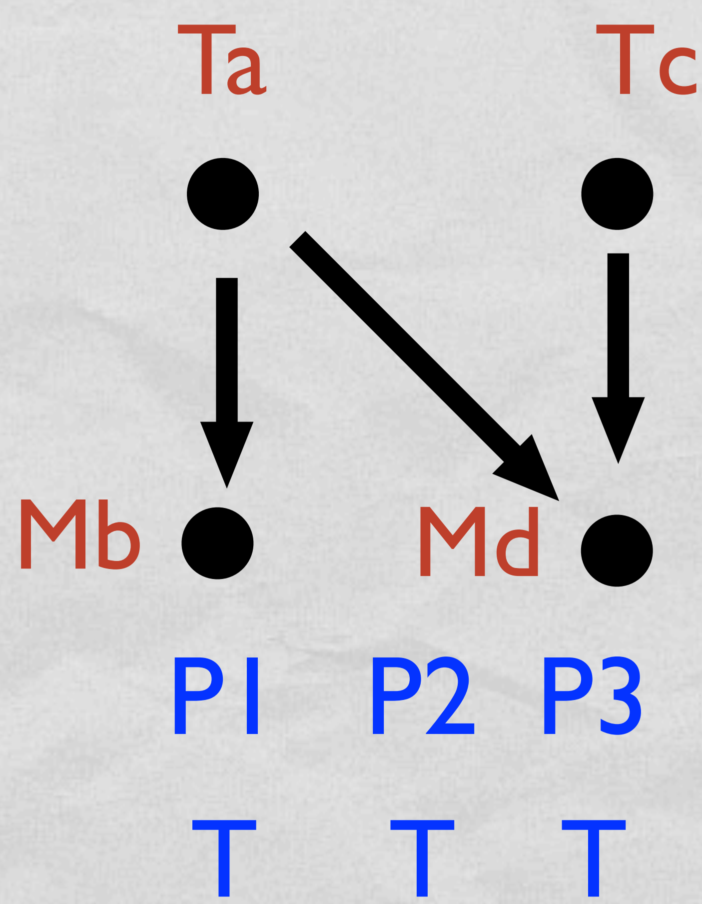
2. $\exists x(M(x) \land T(a) \land A(a,x) \land \forall z((T(z) \land a \neq z) \rightarrow \neg A(x,z)))$

3. $\forall x \forall y((T(x) \land T(y) \land x \neq y) \rightarrow \exists z(M(z) \land A(x,z) \land A(y,z)))$

¬3. $\exists x \exists y(T(x) \land T(y) \land x \neq y \land \forall z(M(z) \rightarrow (\neg A(x,z) \lor \neg A(y,z))))$

Independent?





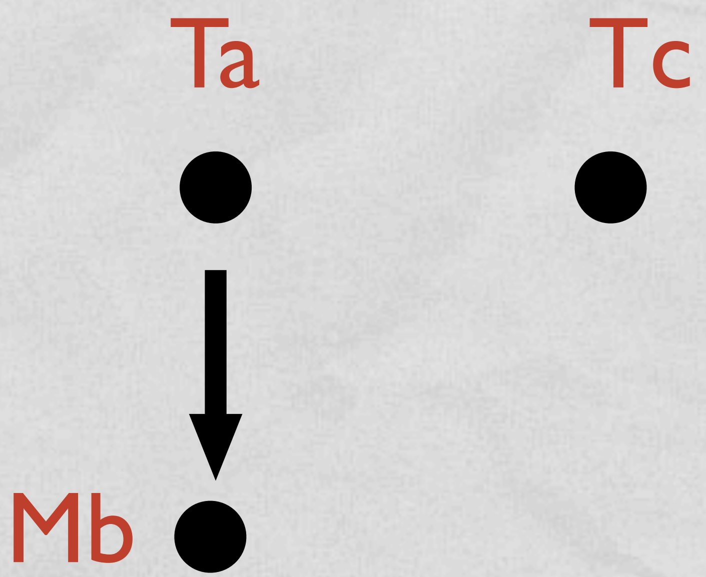
1. $\exists x \exists y(T(x) \wedge T(y) \wedge \exists z(M(z) \wedge A(x,z) \wedge \neg A(y,z)))$

2. $\exists x(M(x) \wedge T(a) \wedge A(a,x) \wedge \forall z((T(z) \wedge a \neq z) \rightarrow \neg A(x,z)))$

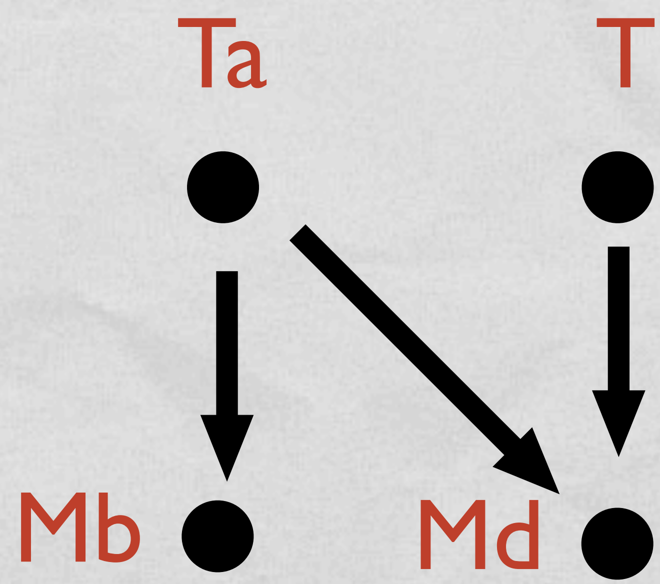
3. $\forall x \forall y((T(x) \wedge T(y) \wedge x \neq y) \rightarrow \exists z(M(z) \wedge A(x,z) \wedge A(y,z)))$

¬3. $\exists x \exists y(T(x) \wedge T(y) \wedge x \neq y \wedge \forall z(M(z) \rightarrow (\neg A(x,z) \vee \neg A(y,z))))$

So P3 is independent of 1+2



P1	P2	P3
T	T	F



P1	P2	P3
T	T	T

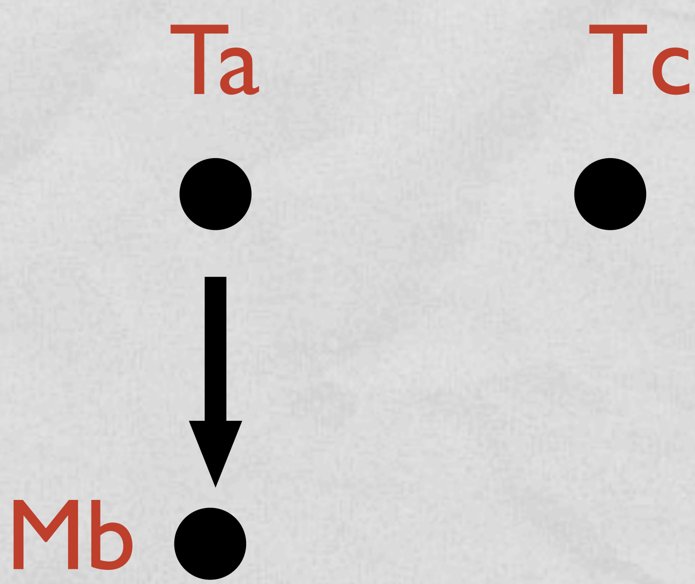
1. $\exists x \exists y(T(x) \wedge T(y) \wedge \exists z(M(z) \wedge A(x,z) \wedge \neg A(y,z)))$

2. $\exists x(M(x) \wedge T(a) \wedge A(a,x) \wedge \forall z((T(z) \wedge a \neq z) \rightarrow \neg A(x,z)))$

¬2. $\forall x((M(x) \wedge T(a) \wedge A(a,x)) \rightarrow \exists z(T(z) \wedge a \neq z \wedge A(x,z)))$

3. $\forall x \forall y((T(x) \wedge T(y) \wedge x \neq y) \rightarrow \exists z(M(z) \wedge A(x,z) \wedge A(y,z)))$

Is P2 independent of 1+3?



P1	P2
T	T

1. $\exists x \exists y(T(x) \wedge T(y) \wedge \exists z(M(z) \wedge A(x,z) \wedge \neg A(y,z)))$

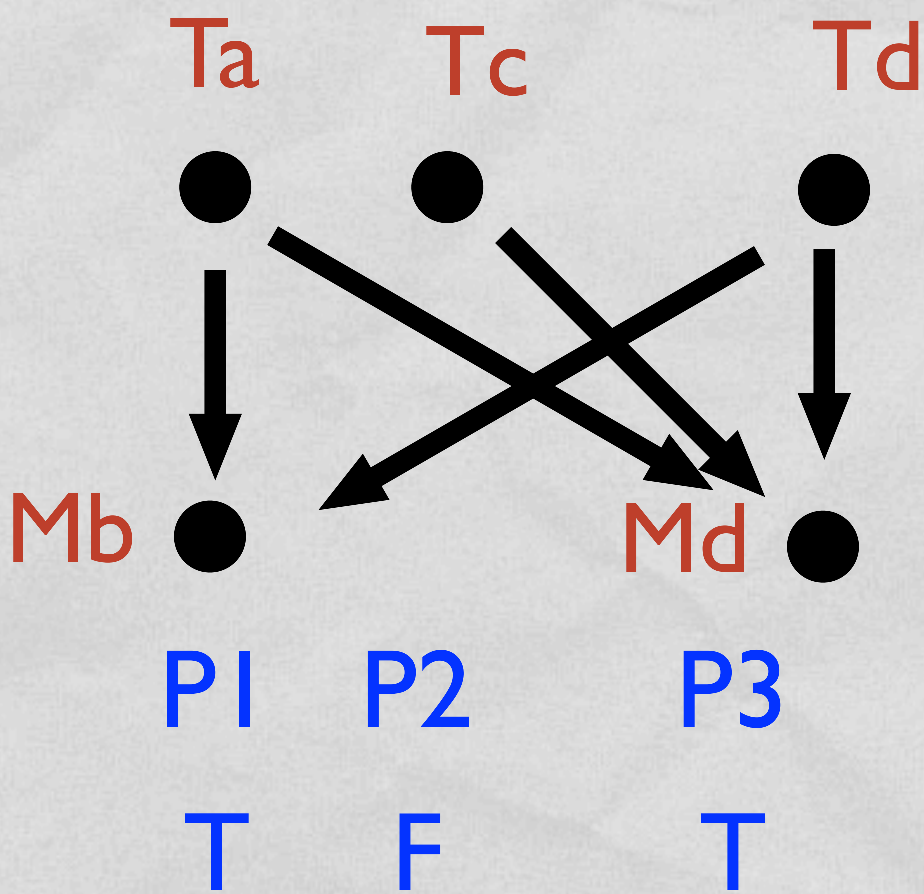
2. $\exists x(M(x) \wedge T(a) \wedge A(a,x) \wedge \forall z((T(z) \wedge a \neq z) \rightarrow \neg A(x,z)))$

¬2. $\forall x((M(x) \wedge T(a) \wedge A(a,x)) \rightarrow \exists z(T(z) \wedge a \neq z \wedge A(x,z)))$

3. $\forall x \forall y((T(x) \wedge T(y) \wedge x \neq y) \rightarrow \exists z(M(z) \wedge A(x,z) \wedge A(y,z)))$

Is P2 independent of 1+3?

Yes



1. $\exists x \exists y(T(x) \land T(y) \land \exists z(M(z) \land A(x,z) \land \neg A(y,z)))$

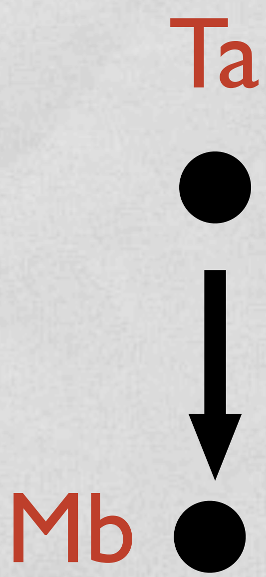
¬1. $\forall x \forall y((T(x) \land T(y)) \rightarrow \forall z(M(z) \rightarrow (A(x,z) \leftrightarrow A(y,z))))$

2. $\exists x(M(x) \land T(a) \land A(a,x) \land \forall z((T(z) \land a \neq z) \rightarrow \neg A(x,z)))$

3. $\forall x \forall y((T(x) \land T(y) \land x \neq y) \rightarrow \exists z(M(z) \land A(x,z) \land A(y,z)))$

Is P1 independent of 2+3?

Yes



P1	P2	P3
F	T	T