PROOF?

Is there anything wrong with the following argument?

Claim: Every natural number is interesting.

Proof: If there are some uninteresting natural numbers, then there is a least number which is uninteresting. Call it ‘a’. But since ‘a’ is the lowest number which is uninteresting, there is something quite interesting about it. So ‘a’ can’t be the first uninteresting number. But ‘a’ was totally arbitrary. So no number can be the first uninteresting number so no number can be uninteresting so every number is interesting.

Using Diagrams for Interpretations

Wednesday, 3 November

Informal Semantics

- FO valid means that any *interpretation* that makes all of the premises true also makes the conclusion true. An interpretation gives the meaning of the constants, functions, and predicates and gives a domain (so we know what 'for all x' means.
- By 'gives the meaning' we just mean gives enough information to make sentences true or false.

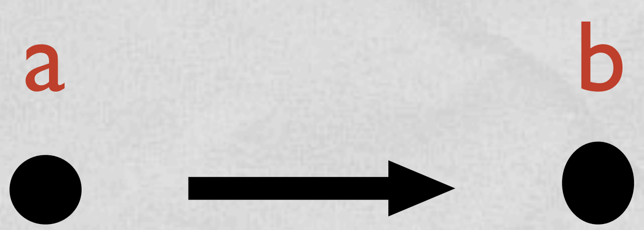
DIAGRAMS

- Two places predicates ("relations") can be very naturally modeled with diagrams.

Here is an example interpretation--

Domain: dots in my picture.

Rxy: x points to y in my picture



R(a,b): True

R(b,a): False

R(a,a): False

R(b,b): False

$\exists x\, R(x,b)$: True

$\forall x\, R(x,b)$: False

$\exists x\, R(x,a)$: False

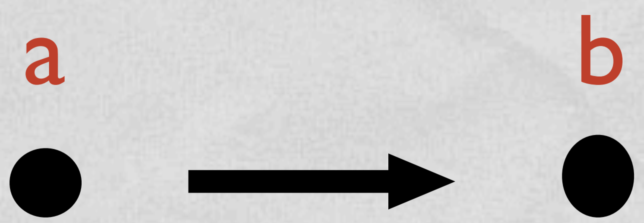
Diagrams

- Two places predicates ("relations") can be very naturally modeled with diagrams.

Here is an example interpretation--

Domain: dots in my picture.

R(x,y): x points to y in my picture



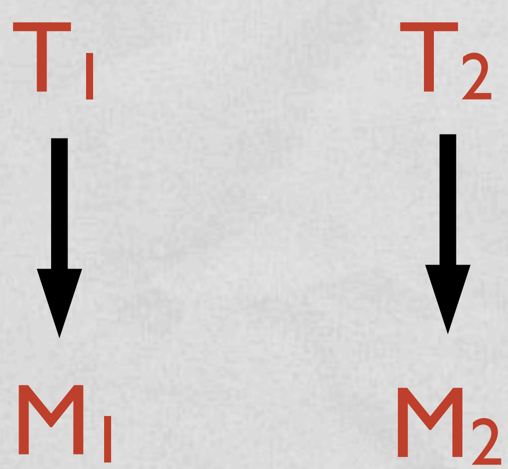
$\forall x(\exists y\, R(x,y) \vee \exists y\, R(y,x))$

True

Of everything, either there is something that it points to, or there is something that points to it

Diagrams

Domain: things in my picture.
A(x,y): x points to y in my picture
T(x): x is labeled ‘T’ in my picture
M(x): x is labeled ‘M’ in my picture



Sometimes it helps to think of English examples. Here teachers attending meetings might be appropriate.

$\forall x(T(x) \rightarrow \exists y(M(y) \wedge A(x,y)))$ True

$\exists x(T(x) \wedge \forall y(M(y) \rightarrow A(x,y)))$ False

Mechanical Verification

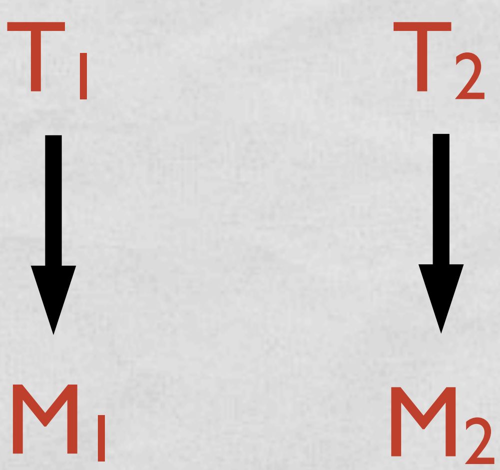
For $\forall x\, P(x)$ to be true in an interpretation, $P(x)$ must be satisfied by every element in the domain.

In an interpretation with a domain of 3 elements (call them a,b,c), $\forall x\, P(x)$ is true if and only if $P(a) \wedge P(b) \wedge P(c)$ is true. [$\exists x$ with $\vee$]

$\forall x(P(x) \rightarrow \exists y(Q(y) \wedge R(x,y)))$ is true if and only if

$P(a) \rightarrow \exists y(Q(y) \wedge R(a,y)) \wedge$
$P(b) \rightarrow \exists y(Q(y) \wedge R(b,y)) \wedge$
$P(c) \rightarrow \exists y(Q(y) \wedge R(c,y))$ is true.

Mechanical Verification



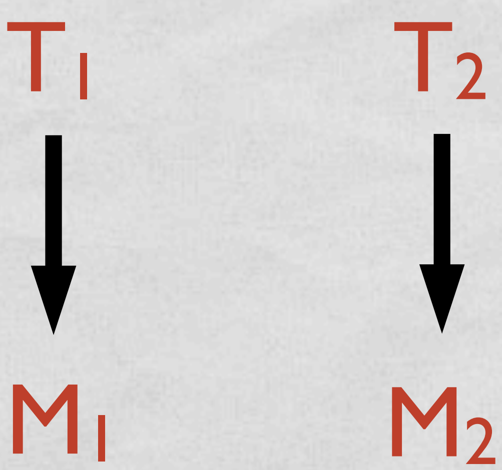
$\forall x(T(x) \rightarrow \exists y(M(y) \wedge A(x,y)))$ True

This universal is true iff a certain conditional is satisfied by all four objects. Since M_1 and M_2 aren't Ts, they satisfy the conditional. So the universal is true just in case the two Ts satisfy it.

$T(t_1) \rightarrow \exists y(M(y) \wedge A(t_1,y))$ True

$T(t_2) \rightarrow \exists y(M(y) \wedge A(t_2,y))$ True

Mechanical Verification



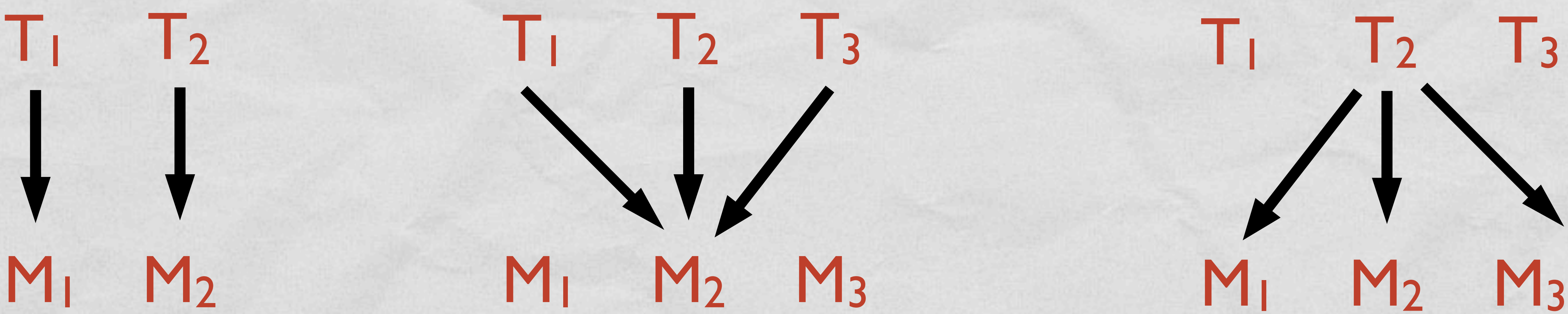
$\exists x(M(x) \land \forall y(T(y) \rightarrow A(y,x)))$ False

This existential is true iff a certain conjunction is satisfied by at least one object. Since T_1 and T_2 aren't Ms, they don't satisfy the conjunction. So the existential is true just in case at least one of the two Ms satisfies it.

$M(m_1) \land \forall y(T(y) \rightarrow A(y, m_1))$ False

$M(m_2) \land \forall y(T(y) \rightarrow A(y, m_2))$ False

EXAMPLES



$\exists x(M(x) \wedge \forall y(T(y) \rightarrow A(y,x)))$ False, True, False

$\forall x(M(x) \rightarrow \neg\forall y(T(y) \rightarrow A(y,x)))$ True, False, True

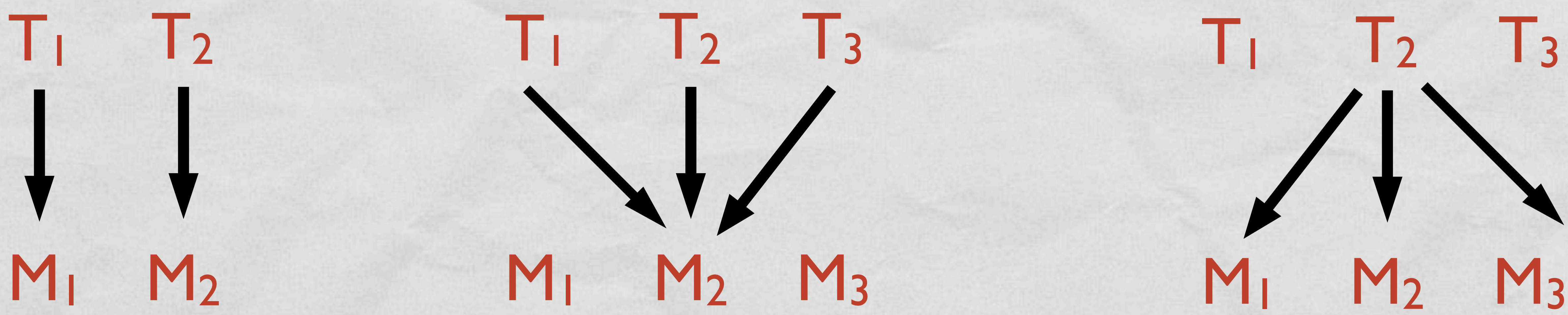
$\forall x(T(x) \rightarrow (\exists y(M(y) \wedge A(x,y)) \wedge \exists y(M(y) \wedge \neg A(x,y))))$

True, True, False

Diagrams and Translations

- Having a translation scheme in mind (teachers attending meetings) is often very helpful to do these problems.
- But don't be wedded to any one scheme - and especially not to a genuine English understanding of that scheme.
- For example, we have to be able to model $T(a) \wedge M(a)$, $A(m_2, t_1)$, and $A(m_2, m_2)$
- In addition, with difficult examples, it takes students a lot of effort to come up with an English sentence and it is often wrong or they get the logic wrong because of their sentence.

Examples



A very natural thing you might want to say about these diagrams essentially involves counting. There is a teacher who went to three meetings for example. For this, you need identity.

Translations with Identity

$\exists x \exists y(P(x) \wedge P(y))$

Both x and y are painters

- but not necessarily different!

$\exists x \exists y(P(x) \wedge P(y) \wedge x \neq y)$

There are at least two painters

$\exists x \exists y(x \neq y)$ There are at least two things in the domain

Translations with Identity

$\exists x \exists y \exists z(x \neq y \wedge y \neq z \wedge x \neq z)$

There are at least three things

$\neg \exists x \exists y \exists z(x \neq y \wedge y \neq z \wedge x \neq z)$

There are NOT at least three things

= There are at most two things (=0,1,or 2)

$\neg \exists x \exists y(P(x) \wedge P(y) \wedge x \neq y)$

= There is at most one painter (0 or 1)

Equivalent Translations

$\neg\exists x\exists y(P(x) \wedge P(y) \wedge x \neq y)$

$\Leftrightarrow \quad \forall x\neg\exists y(P(x) \wedge P(y) \wedge x \neq y)$

$\Leftrightarrow \quad \forall x\forall y\neg(P(x) \wedge P(y) \wedge x \neq y)$

$\Leftrightarrow \quad \forall x\forall y([P(x) \wedge P(y)] \rightarrow x = y)$

= There is at most one painter (0 or 1)

$\forall x\forall y\forall z([P(x) \wedge P(y) \wedge P(z)] \rightarrow (x=y \vee y=z \vee x=z))$

= There is at most two painters (0 or 1 or 2)

Equivalent Translations

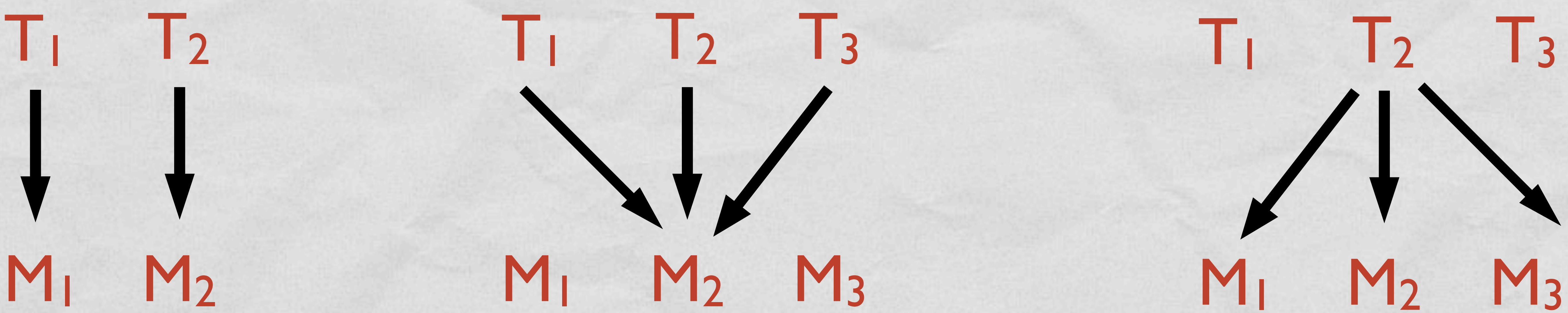
Exactly one = At least one and at most one (not two)

$\exists x\, P(x) \wedge \neg \exists x \exists y (P(x) \wedge P(y) \wedge x \neq y)$

$\Leftrightarrow \quad \exists x\, P(x) \wedge \forall x \forall y ([P(x) \wedge P(y)] \rightarrow x=y)$

$\Leftrightarrow \quad \exists x (P(x) \wedge \forall y (P(y) \rightarrow x=y)$

Examples



$\exists x(M(x) \wedge \exists y \exists z(y \neq z \wedge T(y) \wedge T(z) \wedge A(y,x) \wedge A(z,x)))$

There is a meeting that at least two teachers went to

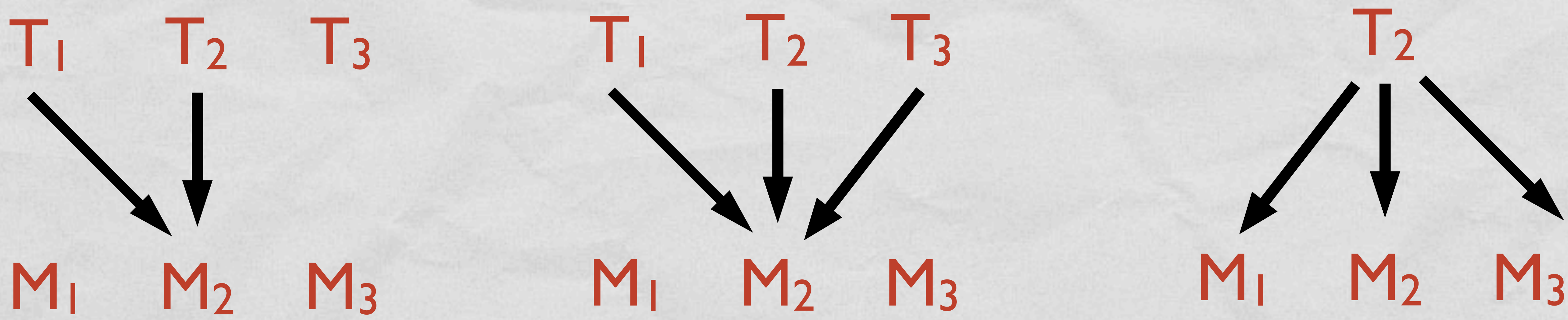
False, True, False

$\forall x \forall y((T(x) \wedge T(y) \wedge x \neq y) \rightarrow \exists z(M(z) \rightarrow (A(x,z) \wedge A(y,z))))$

For every pair of teachers, there is a meeting that they both went to

False, True, False

EQUIVALENT?



$\exists x(M(x) \wedge \exists y \exists z(y \neq z \wedge T(y) \wedge T(z) \wedge A(y,x) \wedge A(z,x)))$

There is a meeting that at least two teachers went to

True, True, False

$\forall x \forall y((T(x) \wedge T(y) \wedge x \neq y) \rightarrow \exists z(M(z) \rightarrow (A(x,z) \wedge A(y,z))))$

For every pair of teachers, there is a meeting that they both went to

False, True, True