Puzzle

In a certain game, players 1 and 2 go back and forth choosing either 1 or 2. If a player brings the sum of all previously chosen numbers to 7 or greater, that player wins. You, player 1, go first. What number should you pick?

Now you can choose numbers 1-n and we play until the sum is x or greater. Do you want to go first or second?

Puzzle Answer

Notice that if it is your turn and the sum so far is 5 or 6, you can win. Therefore if the other player has to choose when the sum is at 4, she will bring the sum to 5 or 6 and so you win. So if you can bring it to 4, you win. So choose 1, then at each step, choose 1 iff they previously chose 2.

If you are choosing from 1-n, you can increase the sum by n+1 each iteration. If x divides n+1 evenly, go second. If it doesn't, go first and bring the sum to exactly z × (n+1) away from x to win [for any z].

Can I win?

A winning strategy is one in which it doesn't matter what your opponent does, you can do something to win.

Domain = {1,2}

Player 1 can win since: $\exists x \forall y \exists z \forall w \exists v (x+y+z+w+x+v=7)$

If Player 2 had a winning strategy it would be true that:

$\neg \exists x \forall y \exists z \forall w \exists v .... \Leftrightarrow$

$\forall x \exists y \forall z \exists w \forall v$ .... (you

Overlapping Scope Proofs and Identity

Friday, 29 October

Testing Validity using Tarski's World

SameRow(a,b)

SameRow(b,a)

Valid or not?

Can't make T, F in Tarski's World. But this clearly depends on the meaning of SameRow. S(a,b) therefore S(b,a) is not FO valid.

What if "SameRow(x,y)" meant RightOf(x,y)?

Testing Validity using Tarski's World

$\forall x(\text{Cube}(x) \lor x=a)$
$\exists x\ \text{Small}(x)$

$\forall x(x=a \rightarrow (\text{Dodec}(x) \lor \text{Small}(x)))$

Valid or not?

Translations with Identity

$\forall x(Cube(x) \lor x=a) \Leftrightarrow \forall x(\neg Cube(x) \rightarrow x=a)$

If like this then you are =a

a is the only non-cube (if there is one)

$\forall x(x=a \rightarrow (Dodec(x) \lor Small(x)))$

If you are =a then you are like this

a is either a dodec or is small

Translations with Identity

$\forall x(x{=}a \rightarrow P(x)) \quad \Leftrightarrow \quad P(a)$

$\exists x(x{=}a \wedge P(x)) \quad \Leftrightarrow \quad P(a)$

therefore

$\vdash \forall x(x{=}a \rightarrow P(x)) \leftrightarrow \exists x(x{=}a \wedge P(x))$

1. $\forall x(x=a \rightarrow P(x))$

2. $a=a \rightarrow P(a)$ — $\forall$ Elim 1

3. $a=a$ — = Intro

4. $a=a \land P(a)$ — Taut Con 2,3

$\exists x(x=a \land P(x))$

$\exists x(x=a \land P(x))$

$\forall x(x=a \rightarrow P(x))$

$\forall x(x=a \rightarrow P(x)) \leftrightarrow \exists x(x=a \land P(x))$ — $\leftrightarrow$ Intro

1. $\forall x(x=a \rightarrow P(x))$

2. $a=a \rightarrow P(a)$ — $\forall$ Elim 1

3. $a=a$ — = Intro

4. $a=a \land P(a)$ — Taut Con 2,3

5. $\exists x(x=a \land P(x))$ — $\exists$ Intro 4

6. $\exists x(x=a \land P(x))$

$\forall x(x=a \rightarrow P(x))$

$\forall x(x=a \rightarrow P(x)) \leftrightarrow \exists x(x=a \land P(x))$ — $\leftrightarrow$ Intro

6. $\exists x(x=a \land P(x))$

7. $\boxed{b}$ $b=a \land P(b)$

8. $\boxed{c}$

9. $c=a$

10. $b=c \land P(b)$ — $=$ Elim 7,9

11. $b=c$ — $\land$ Elim 10

12. $P(b)$ — $\land$ Elim 10

$P(c)$

$c=a \rightarrow P(c)$ — $\rightarrow$ Intro

$\forall x(x=a \rightarrow P(x))$ — $\forall$ Intro

$\forall x(x=a \rightarrow P(x))$ — $\exists$ Elim

$\forall x(x=a \rightarrow P(x)) \leftrightarrow \exists x(x=a \land P(x))$ — $\leftrightarrow$ Intro

6. $\exists x(x=a \land P(x))$

7. $\boxed{b}$ $b=a \land P(b)$

8. $\boxed{c}$

9. $c=a$

10. $b=c \land P(b)$ — $=$ Elim 7,9

11. $b=c$ — $\land$ Elim 10

12. $P(b)$ — $\land$ Elim 10

13. $P(c)$ — $=$ Elim 11,12

14. $c=a \rightarrow P(c)$ — $\rightarrow$ Intro 9-13

15. $\forall x(x=a \rightarrow P(x))$ — $\forall$ Intro 8-14

16. $\forall x(x=a \rightarrow P(x))$ — $\exists$ Elim 6,7-15

17. $\forall x(x=a \rightarrow P(x)) \leftrightarrow \exists x(x=a \land P(x))$ — $\leftrightarrow$ Intro 1-5, 6-116

1. $\forall x \exists y (P(x) \land Q(y))$	
2. $\exists y (P(a) \land Q(y))$	$\forall$ Elim 1
3. [b] $P(a) \land Q(b)$	
4. [c]	
5. $\exists y (P(c) \land Q(y))$	$\forall$ Elim 1
6. [d] $P(c) \land Q(d)$	
$P(c) \land Q(b)$	
$P(c) \land Q(b)$	$\exists$ Elim
$\forall x (P(x) \land Q(b))$	$\forall$ Intro
$\exists y \forall x (P(x) \land Q(y))$	$\exists$ Intro
$\exists y \forall x (P(x) \land Q(y))$	$\exists$ Elim

1. $\forall x \exists y(P(x) \land Q(y))$	
2. $\exists y(P(a) \land Q(y))$	$\forall$ Elim 1
3. $\boxed{b}$ $P(a) \land Q(b)$	
4. $\boxed{c}$	
5. $\exists y(P(c) \land Q(y))$	$\forall$ Elim 1
6. $\boxed{d}$ $P(c) \land Q(d)$	
7. $P(c) \land Q(b)$	Taut Con 3,6
8. $P(c) \land Q(b)$	$\exists$ Elim 5,6-7
9. $\forall x(P(x) \land Q(b))$	$\forall$ Intro 4-8
10. $\exists y \forall x(P(x) \land Q(y))$	$\exists$ Intro 9
11. $\exists y \forall x(P(x) \land Q(y))$	$\exists$ Elim 2,3-10

Translations with Identity

$\exists x \exists y(P(x) \land P(y))$

Both x and y are painters

- but not necessarily different!

$\exists x \exists y(P(x) \land P(y) \land x \neq y)$

There are at least two painters

$\exists x \exists y(x \neq y)$ There are at least two things in the domain

Translations with Identity

$\exists x \exists y \exists z(x \neq y \wedge y \neq z \wedge x \neq z)$

There are at least three things

$\neg \exists x \exists y \exists z(x \neq y \wedge y \neq z \wedge x \neq z)$

There are NOT at least three things

= There are at most two things (=0,1,or 2)

$\neg \exists x \exists y(P(x) \wedge P(y) \wedge x \neq y)$

= There is at most one painter (0 or 1)

Equivalent Translations

$\neg\exists x\exists y(P(x) \wedge P(y) \wedge x \neq y)$

$\Leftrightarrow \quad \forall x\neg\exists y(P(x) \wedge P(y) \wedge x \neq y)$

$\Leftrightarrow \quad \forall x\forall y\neg(P(x) \wedge P(y) \wedge x \neq y)$

$\Leftrightarrow \quad \forall x\forall y([P(x) \wedge P(y)] \rightarrow x = y)$

= There is at most one painter (0 or 1)

$\forall x\forall y\forall z([P(x) \wedge P(y) \wedge P(z)] \rightarrow (x=y \vee y=z \vee x=z))$

= There is at most two painters (0 or 1 or 2)

Equivalent Translations

Exactly one = At least one and at most one (not two)

$\exists x\, P(x) \wedge \neg \exists x \exists y (P(x) \wedge P(y) \wedge x \neq y)$

$\Leftrightarrow \quad \exists x\, P(x) \wedge \forall x \forall y ([P(x) \wedge P(y)] \rightarrow x = y)$

$\Leftrightarrow \quad \exists x (P(x) \wedge \forall y (P(y) \rightarrow x = y)$

$\Leftrightarrow \quad \forall y (P(y) \leftrightarrow x = y)$