Puzzle

You heard a rumor that there is gold on the island of knights and knaves and you are sure that each inhabitant knows the truth about this. You meet a random inhabitant who you know is a knight or a knave, but you don't know which. How can you find out the truth about whether there is gold or not with a single 'yes' or 'no' question?

HINT: Notice that if you ask "P" and inhabitant x says "yes" then you know that [Knight(x) $\leftrightarrow$ P] is true and

[Knight(x) $\leftrightarrow$ $\neg$P] is true if they say "no".

Equivalence in FOL

Wednesday, 20 October

First-Order Validity and Consequence

Propositional Logic	First-Order Logic	Basic Notion
Tautology	FO Valid	Logical Truth
Tautological Consequence	FO Consequence	Logical Consequence
Tautological Equivalence	FO Equivalence	Logical Equivalence

First-Order Equivalence

- Contraposition: $P \rightarrow Q$ is taut equivalent to $\neg Q \rightarrow \neg P$
- This is true for any FOL sentences so $\neg\exists x Cube(x) \rightarrow \exists y\, Small(y)$ is taut equivalent to $\neg\exists y\, Small(y) \rightarrow \neg\neg\exists x Cube(x)$
- $\forall x(Cube(x) \rightarrow Small(x))$ and $\forall x(\neg Small(x) \rightarrow \neg Cube(x))$ are FOL equivalent, but not taut equivalent

First-Order Equivalence

Substitution of bound variables

$\forall x\ P(x) \Leftrightarrow \forall y\ P(y)$

$\exists x\ P(x) \Leftrightarrow \exists y\ P(y)$

1. ∃x Cube(x)	
2. [a] Cube(a)	(for ∃ Elim)
3. ∃y Cube(y)	∃ Intro 2
4. ∃y Cube(y)	∃ Elim 1,2-3

EQUIVALENCES FOR QUANTIFIERS

- $\forall x(P(x) \wedge Q(x)) \Leftrightarrow \forall x\, P(x) \wedge \forall x\, Q(x)$
- $\forall x(P(x) \vee Q(x)) \not\Leftrightarrow \forall x\, P(x) \vee \forall x\, Q(x)$

$\rightarrow$ no,

but $\leftarrow$ yes

- $\exists x(P(x) \vee Q(x)) \Leftrightarrow \exists x\, P(x) \vee \exists x\, Q(x)$
- $\exists x(P(x) \wedge Q(x)) \not\Leftrightarrow \exists x\, P(x) \wedge \exists x\, Q(x)$

$\rightarrow$ yes,

but $\leftarrow$ no

Pushing Quantifiers Around

- $\forall x(P(x) \vee Q(x)) \not\Leftrightarrow \forall x\, P(x) \vee \forall x\, Q(x)$
- $\forall x(P \vee Q(x)) \Leftrightarrow P \vee \forall x\, Q(x)$
- $\exists x(P(x) \wedge Q(x)) \not\Leftrightarrow \exists x\, P(x) \wedge \exists x\, Q(x)$
- $\exists x(P \wedge Q(x)) \Leftrightarrow P \wedge \exists x\, Q(x)$

Negated Quantifiers

- $\forall x P(x)$ is like a big conjunction.
- $\neg\forall x P(x)$ is like the negation of a big conjunction.

By DeMorgan's like thinking....

- $\neg\forall x P(x) \Leftrightarrow \exists x \neg P(x)$ (a big disjunction of negations)

By the same thought....

- $\neg\exists x P(x) \Leftrightarrow \forall x \neg P(x)$

Negated Quantifiers

1. ¬∃x P(x)

2. a

3. P(a) (for ¬ Intro)

4. ∃x P(x) ∃ Intro 3

⊥ ⊥Intro

¬P(a) ¬ Intro

∀x ¬P(x) ∀ Intro

Negated Quantifiers

1. ¬∃x P(x)
2. a
3. P(a) (for ¬ Intro)
4. ∃x P(x) ∃ Intro 3
5. ⊥ ⊥Intro 1,4
6. ¬P(a) ¬ Intro 3-5
7. ∀x ¬P(x) ∀ Intro 2-6

NEGATED QUANTIFIERS

1. $\exists x\ \neg P(x)$

2. $\boxed{a}$ $\neg P(a)$ (for $\exists$ Elim)

3. $\forall x\ P(x)$ (for $\neg$ Intro)

4. $P(a)$ $\forall$ Elim 3

$\bot$ $\bot$ Intro

$\neg \forall x\ P(x)$ $\neg$ Intro

$\neg \forall x\ P(x)$ $\exists$ Elim

Negated Quantifiers

1. $\exists x\ \neg P(x)$

2. $\boxed{a}$ $\neg P(a)$ (for $\exists$ Elim)

3. $\forall x\ P(x)$ (for $\neg$ Intro)

4. $P(a)$ $\forall$ Elim 3

5. $\bot$ $\bot$Intro 2,4

6. $\neg\forall x\ P(x)$ $\neg$ Intro 3-5

7. $\neg\forall x\ P(x)$ $\exists$ Elim 1,2-6

Quantifiers and Conditionals

- $\forall x(P(x) \rightarrow Q(x)) \;\; \begin{matrix} \rightarrow \\ \not\leftarrow \end{matrix} \;\; \forall x\, P(x) \rightarrow \forall x\, Q(x)$

It does work the other way for existentials, but that is really hard to think about...

- $\exists x(P(x) \rightarrow Q(x)) \;\; \begin{matrix} \not\rightarrow \\ \leftarrow \end{matrix} \;\; \exists x\, P(x) \rightarrow \exists x\, Q(x)$

Example of FOL Equivalence

$\forall x(Cube(x) \rightarrow (Small(x) \land Tet(b))) \Leftrightarrow$

$\forall x[(Cube(x) \rightarrow Tet(b)) \land (Cube(x) \rightarrow Small(x))] \Leftrightarrow$

$\forall x(Cube(x) \rightarrow Tet(b)) \land \forall x(Cube(x) \rightarrow Small(x)) \Leftrightarrow$

$\forall x(\neg Cube(x) \lor Tet(b)) \land \forall x(Cube(x) \rightarrow Small(x)) \Leftrightarrow$

$(\forall x \neg Cube(x) \lor Tet(b)) \land \forall x(Cube(x) \rightarrow Small(x)) \Leftrightarrow$

$(\neg \exists x Cube(x) \lor Tet(b)) \land \forall x(Cube(x) \rightarrow Small(x))$

Quantifiers and Conditionals

$\forall x(P \lor Q(x)) \Leftrightarrow P \lor \forall x\, Q(x)$

and

$\exists x(P \lor Q(x)) \Leftrightarrow P \lor \exists x\, Q(x)$

so

$\forall x(P \to Q(x)) \Leftrightarrow P \to \forall x Q(x)$

and

$\exists x(P \to Q(x)) \Leftrightarrow P \to \exists x\, Q(x)$

but

$\forall x(P(x) \to Q) \Leftrightarrow \exists x\, P(x) \to Q$

and

$\exists x(P(x) \to Q) \Leftrightarrow \forall x\, P(x) \to Q$

since

$\forall x(\neg P(x) \lor Q) \Leftrightarrow \neg \exists x\, P(x) \lor Q$

and

$\exists x(\neg P(x) \lor Q) \Leftrightarrow \neg \forall x\, P(x) \lor Q$

Quantifiers and Conditionals

If anyone goes to the party, then Bob will be happy

$\exists x \, Party(x) \rightarrow Happy(bob)$

It is true of everyone that if they go to the party, then Bob will be happy

$\forall x \, (Party(x) \rightarrow Happy(bob))$

If Anyone Goes...

1. $\exists x$ Party(x) → Happy(bob)
2. a (for ∀ Intro)
3. Party(a) (for → Intro)
4. $\exists x$ Party(x) ∃ Intro 3

Happy(bob)

Party(a) → Happy(bob) → Intro

∀x(Party(x) → Happy(bob)) ∀ Intro

If Anyone Goes...

1. $\exists x$ Party(x) → Happy(bob)
2. a (for $\forall$ Intro)
3. Party(a) (for → Intro)
4. $\exists x$ Party(x) $\exists$ Intro 3
5. Happy(bob) →Elim 1,4
6. Party(a) → Happy(bob) → Intro 3-5
7. $\forall x$(Party(x) → Happy(bob) $\forall$ Intro 6