PUZZLE

You are an inhabitant of the island of knights and knaves. Everyone on the island is either rich or poor.

1) You are a rich knave. Prove it with one statement.
2) You are a rich knight. Prove it with one statement.
3) Now there are normals on the island too. Prove you are normal.
4) Could anyone prove they aren't normal?

FORMAL PROOFS WITH QUANTIFIERS

Monday, 18 October

Universal Elimination

- For any variable x, any wff P(x), and any constant c, from $\forall x\ P(x)$ we can infer P(c).
- Note: the constant c could even have been used in the proof already.

1.	$\forall y(Cube(y) \rightarrow \exists z(Adjoins(z,y))$	
2.	$Cube(a) \rightarrow \exists z(Adjoins(z,a)$	$\forall$ Elim: 1

Universal Introduction

- For a constant c naming an arbitrary object, any variable x, and any wff P(x), if we show in a subproof that P(c), we can conclude that ∀x P(x).
- Note: the constant c must be new. The step will only work if c only occurs within the subproof.

7. [c]

...

12. Small(c) ∧ BackOf(c,d)

13. ∀y(Small(y) ∧ BackOf(y,d)) ∀ Intro: 7-12

1. P(a) → ¬∀x Q(x)

2. ∀x(Q(x) ∨ R(x))

3. ∀x ¬R(x)

4. ∀y P(y)

5. P(a) ∀ Elim 4

6. ¬∀x Q(x) → Elim 1,5

7. b

8. Q(b) ∨ R(b) ∀ Elim 2

9. ¬R(b) ∀ Elim 3

Q(b)

∀x Q(x) ∀ Intro

⊥ ⊥ Intro

¬∀y P(y) ¬ Intro

∀x ¬R(x) → ¬∀y P(y)) → Intro

only way to use line 6

1.	$P(a) \rightarrow \neg \forall x\, Q(x)$	
2.	$\forall x (Q(x) \vee R(x))$	
3.	$\forall x\, \neg R(x)$	
4.	$\forall y\, P(y)$	
5.	$P(a)$	$\forall$ Elim 4
6.	$\neg \forall x\, Q(x)$	$\rightarrow$ Elim 1,5
7.	$\boxed{b}$	
8.	$Q(b) \vee R(b)$	$\forall$ Elim 2
9.	$\neg R(b)$	$\forall$ Elim 3
10.	$Q(b)$	Taut Con 8,9
11.	$\forall x\, Q(x)$	$\forall$ Intro 10
12.	$\bot$	$\bot$ Intro 6,11
13.	$\neg \forall y\, P(y)$	$\neg$ Intro 7-12
14.	$\forall x\, \neg R(x) \rightarrow \neg \forall y\, P(y))$	$\rightarrow$ Intro 3-13

Existential Introduction

- For any variable x, any wff P(x) and any constant c, if we show that P(c), we can conclude that ∃x P(x).
- Note: the constant c could even have been used in the proof already. - ∀ Elim and ∃ Intro are 'anything goes' rules.

1. P(c)	
2. ∃x P(x)	∃ Intro: 1

Existential Elimination

- Existential elimination is like proof by cases, but with only one case representing an infinite number of cases.
- For a constant c naming an arbitrary object, any variable x, and any wff P(x), if we know that ∃x P(x), and we assume P(c) and show in a subproof that Q follows (and Q does not contain 'c'), we can conclude that Q must be true (outside the subproof).
- Note: the constant c must be new. The step will only work if c only occurs within the subproof.

Existential Elimination

1. ∃x P(x)

   2. [c] P(c)

   ...

   j. Q

k. Q ∃ Elim: 1,2-j

Existential Elimination

1. ∃x Cube(x)
2. ∀x (Small(x) → ¬Cube(x))

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3. [a] Cube(a)   (for ∃ Elim)

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4. Small(a) → ¬Cube(a)   ∀ Elim 2
5. ¬Small(a)   Taut Con 3,4

∃x ¬Small(x)

∃x ¬Small(x)   ∃ Elim

Existential Elimination

1. ∃x Cube(x)
2. ∀x (Small(x) → ¬Cube(x))

---

3. [a] Cube(a)   (for ∃ Elim)

---

4. Small(a) → ¬Cube(a)   ∀ Elim 2
5. ¬Small(a)   Taut Con 3,4

6. ∃x ¬Small(x)   ∃ Intro 5

7. ∃x ¬Small(x)   ∃ Elim 1,3-6

Existential Elimination

1. $\exists x(Cube(x) \wedge RightOf(x,a))$
2. $\forall x\ (RightOf(x,a) \rightarrow SameSize(a,x))$

3. $\boxed{b}$ $Cube(b) \wedge RightOf(b,a)$ (for ∃ Elim)

4. $RightOf(b,a) \rightarrow SameSize(a,b)$ ∀ Elim 2

5. $Cube(b) \wedge SameSize(a,b)$ Taut Con 3,4

6. $\exists x(Cube(x) \wedge SameSize(a,x))$

7. $\exists x(Cube(x) \wedge SameSize(a,x))$ ∃ Elim from 1

Existential Elimination

1. $\exists x(Cube(x) \land RightOf(x,a))$
2. $\forall x\ (RightOf(x,a) \rightarrow SameSize(a,x))$

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3. $\boxed{b}$ $Cube(b) \land RightOf(b,a)$ (for ∃ Elim)

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4. $RightOf(b,a) \rightarrow SameSize(a,b)$ ∀ Elim 2
5. $Cube(b) \land SameSize(a,b)$ Taut Con 3,4
6. $\exists x(Cube(x) \land SameSize(a,x))$ ∃ Intro 5
7. $\exists x(Cube(x) \land SameSize(a,x))$ ∃ Elim 1,3-6

1. $\forall x((Q(x) \lor R(x)) \to P(x))$

2. $\exists x\, Q(x)$

3. $\exists x\, \neg P(x)$

4. [a] $Q(a)$ (for ∃ Elim from 2)

5. $(Q(a) \lor R(a)) \to P(a)$ ∀ Elim 1

6. $P(a)$ Taut Con 4,5

7. $\exists y\, P(y)$ ∃ Intro 6

8. [b] $\neg P(b)$ (for ∃ Elim from 3)

9. $(Q(b) \lor R(b)) \to P(b)$ ∀ Elim 1

10. $\neg R(b)$ Taut Con 8,9

11. $\exists x\, \neg R(x)$ ∃ Intro 10

$\exists x\, \neg R(x) \land \exists y\, P(y)$

$\exists x\, \neg R(x) \land \exists y\, P(y)$ ∃ Elim from 3

$\exists x\, \neg R(x) \land \exists y\, P(y)$ ∃ Elim from 2

1. $\forall x((Q(x) \lor R(x)) \to P(x))$

2. $\exists x\, Q(x)$

3. $\exists x\, \neg P(x)$

4. $\boxed{a}$ $Q(a)$ (for ∃ Elim from 2)

5. $(Q(a) \lor R(a)) \to P(a)$ ∀ Elim 1

6. $P(a)$ Taut Con 4,5

7. $\exists y\, P(y)$ ∃ Intro 6

8. $\boxed{b}$ $\neg P(b)$ (for ∃ Elim from 3)

9. $(Q(b) \lor R(b)) \to P(b)$ ∀ Elim 1

10. $\neg R(b)$ Taut Con 8,9

11. $\exists x\, \neg R(x)$ ∃ Intro 10

12. $\exists x\, \neg R(x) \land \exists y\, P(y)$ ∧ Intro 7,11

13. $\exists x\, \neg R(x) \land \exists y\, P(y)$ ∃ Elim 3,8-12

14. $\exists x\, \neg R(x) \land \exists y\, P(y)$ ∃ Elim 2,4-13

The Proof System $\mathcal{F}$

- We now know all of the rules of the natural deduction system $\mathcal{F}$. These are the rules of $\mathcal{F}_T$ plus ∀Elim, ∀Intro, ∃Elim, ∃Intro, =Elim, and =Intro.

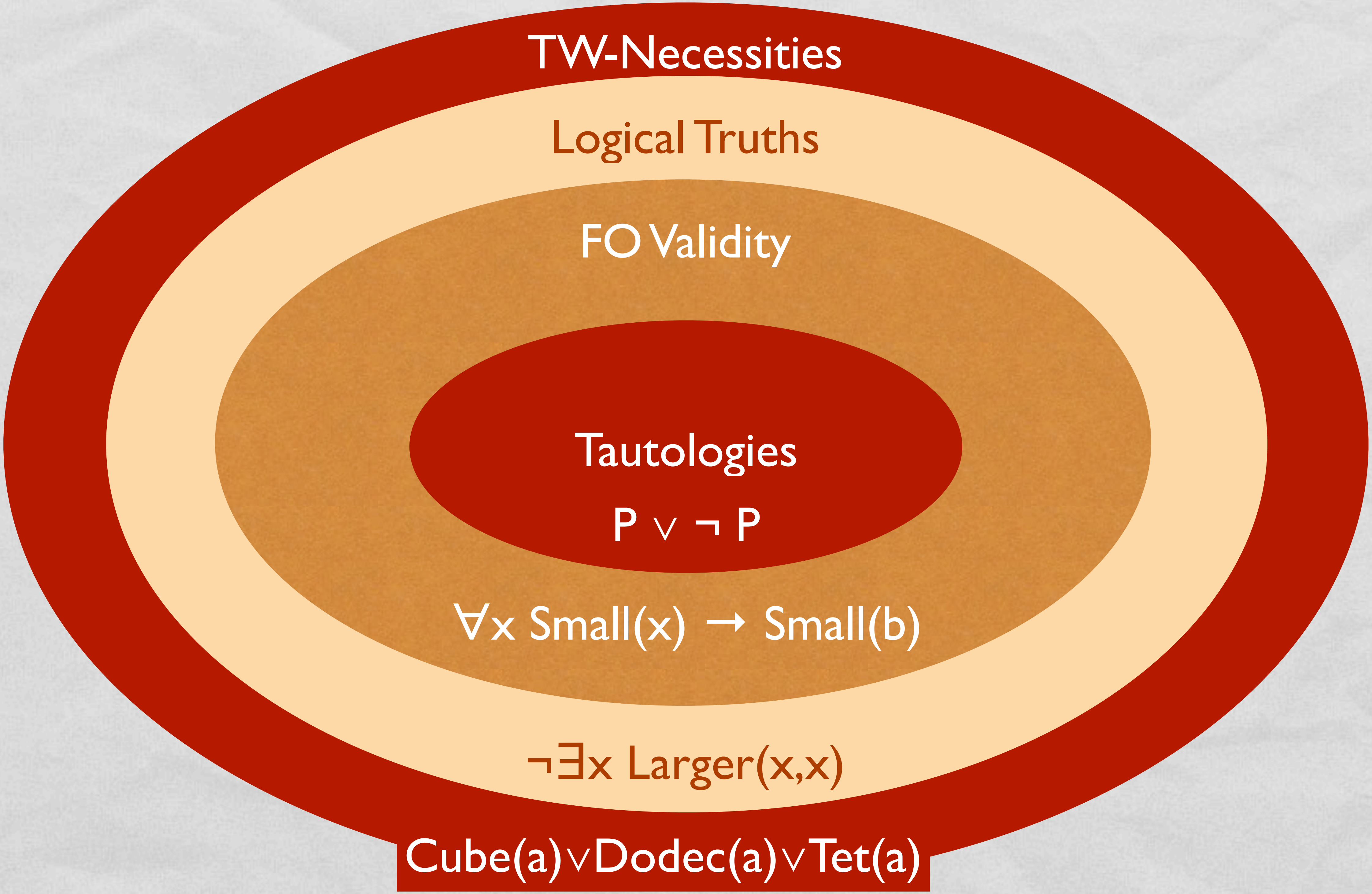
- It turns out that this system is Sound and Complete with respect to the semantics for First Order Logic.

- This system can prove the validity of all and only arguments that are valid in virtue of the propositional connectives, the quantifier symbols, and identity. The meanings of the predicates (other than identity) don't count (so SameSize(a,a) isn't a theorem of $\mathcal{F}$).

First-Order Validity and Consequence

Propositional Logic	First-Order Logic	Basic Notion
Tautology	FO Valid	Logical Truth
Tautological Consequence	FO Consequence	Logical Consequence
Tautological Equivalence	FO Equivalence	Logical Equivalence

First-Order Validity and Consequence



First-Order Equivalence

- Contraposition: $P \rightarrow Q$ taut equivalent to $\neg Q \rightarrow \neg P$
- This is true for any FOL sentences so $\neg\exists x Cube(x) \rightarrow \exists y\ Small(y)$ is taut equivalent to $\neg\exists y\ Small(y) \rightarrow \neg\neg\exists x Cube(x)$
- $\forall x(Cube(x) \rightarrow Small(x))$ and $\forall x(\neg Small(x) \rightarrow \neg Cube(x))$ are FOL equivalent, but not taut equivalent

Equivalences for Quantifiers

- $\forall x(P(x) \land Q(x)) \Leftrightarrow \forall x\, P(x) \land \forall x\, Q(x)$
- $\forall x(P(x) \lor Q(x)) \not\Leftrightarrow \forall x\, P(x) \lor \forall x\, Q(x)$

  $\rightarrow$ no,
  but $\leftarrow$ yes

- $\exists x(P(x) \lor Q(x)) \Leftrightarrow \exists x\, P(x) \lor \exists x\, Q(x)$
- $\exists x(P(x) \land Q(x)) \not\Leftrightarrow \exists x\, P(x) \land \exists x\, Q(x)$

  $\rightarrow$ yes,
  but $\leftarrow$ no