PUZZLE

You know that at least one (possibly more) of A,B,C are involved in a bank robbery and you know no one else was involved. You also know:

If A is guilty and B is innocent, then C is guilty
C never works alone
A never works with C

Can you safely infer the innocence or guilt of any of them?

Formal Proofs with Quantifiers

Friday, 15 October

Complex Predicates

There is a large cube to the left of *b*

$\exists x(L(x) \wedge C(x) \wedge LO(x,b))$

There is a cube to the left of *b* which is in the same row as *c*

$\exists y(C(y) \wedge LO(y,b) \wedge SR(y,c))$

b is in the same row as a large cube

$\exists x(L(x) \wedge C(x) \wedge SR(b,x))$

Complex Predicates

All Ps are Qs	$\forall x(P(x) \rightarrow Q(x))$
All Ps that are also Rs are Qs	$\forall x([P(x) \land R(x)] \rightarrow Q(x))$
All cubes are to the right of *a*	$\forall x(Cubes(x) \rightarrow RightOf(x,a))$
All small cubes are to the right of *a*	$\forall z([Small(z) \land Cube(z)] \rightarrow RightOf(z,a))$

Complex Predicates

Every tall boy is a happy painter

$\forall x([T(x) \wedge B(x)] \rightarrow [H(x) \wedge P(x)])$

Not every cube in the same row as *b* is medium

$\neg \forall w([C(w) \wedge SR(w,b)] \rightarrow M(w))$

No cubes in the same row as *b* are medium

$\forall x([C(x) \wedge SR(x,b)] \rightarrow \neg M(x))$

Every cube that is either small or medium is smaller than *b*

$\forall x([C(x) \wedge (S(x) \vee M(x))] \rightarrow Sm(x,b))$

OTHER FORMS

If every block is a cube, then none are dodecs

$\forall x C(x) \rightarrow \forall y \neg D(y)$

Every cube is small if and only if it isn't large

$\forall x(C(x) \rightarrow (S(x) \leftrightarrow \neg L(x)))$

Every cube is either small or medium

$\forall x(C(x) \rightarrow (S(x) \lor M(x)))$

Either every cube is small or every cube is medium

$\forall x(C(x) \rightarrow S(x)) \lor \forall x(C(x) \rightarrow M(x))$

Satisfaction - again

$\forall x(x=a \rightarrow Tet(x))$ **T**

$\exists x(x \neq a \wedge Small(x) \wedge Tet(x))$ **F**

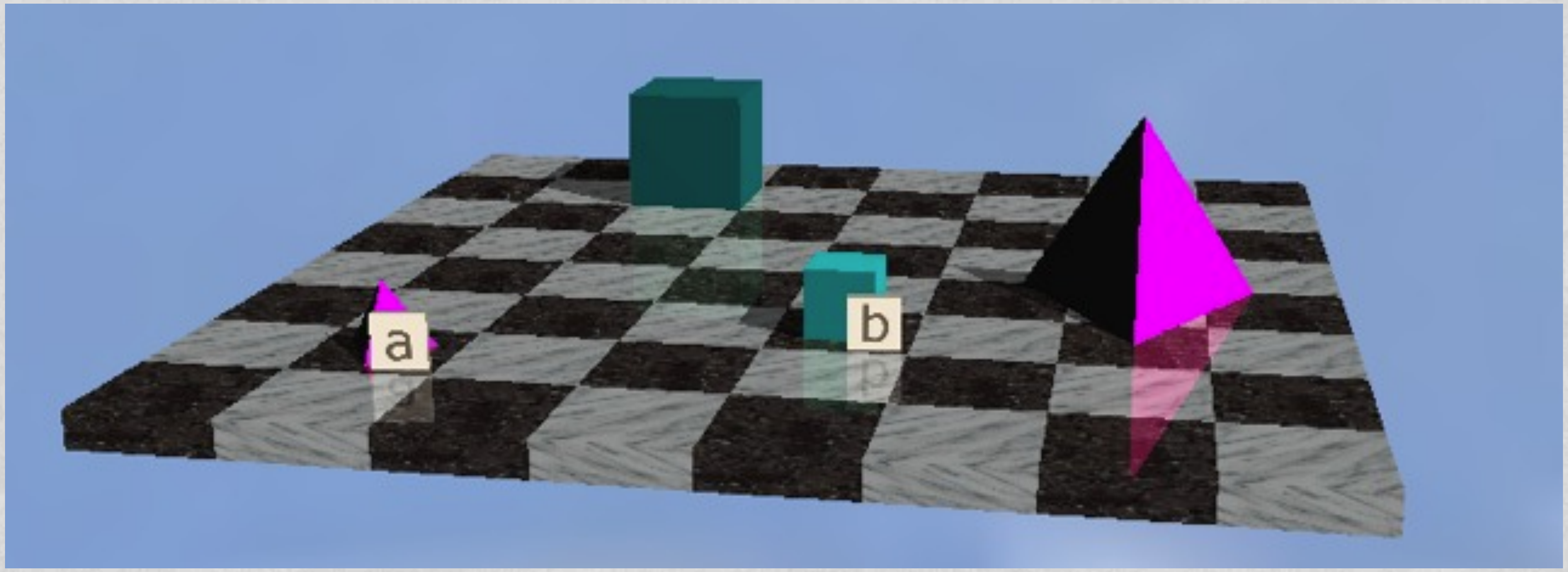
$\forall x((Small(x) \wedge Cube(x)) \rightarrow RightOf(x,a))$ **T**

$\forall x\, RightOf(x,a)$ **F**

$\forall x(Tet(x) \rightarrow (FrontOf(x,b) \rightarrow Small(x)))$ **T**

$\exists x\, SameSize(x,a) \rightarrow x=b$

Not a sentence



Quantifiers and Tautologies

- Remember that tautological consequence, tautological necessity, tautological equivalence, etc., depend on the Boolean connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$). We can evaluate tautological notions with truth tables.
- Quantified sentences are sentences too - so they can be tautologies, can be tf-equivalent to other sentences, can tf-entail sentences, etc.

Quantifiers and Tautologies

- $P \lor \neg P$ is a tautology.
- $\exists x\ Cube(x) \lor \exists x\ \neg Cube(x)$ is not.
- $\forall x\ Cube(x) \lor \forall x\ \neg Cube(x)$ isn't either.
- But $\forall x\ Cube(x) \lor \neg\forall x\ Cube(x)$ is a tautology.
- Let $P = \forall x\ Cube(x)$. Then $\forall x Cube(x) \lor \neg\forall x Cube(x)$ is just $P \lor \neg P$.

Truth-Functional Form

- The truth-functional form algorithm can be used to distinguish tautologies and tautological consequence from logical truths and logical consequences that depend upon the quantifiers, identity, or predicate meanings.
- First, annotate the sentence: underline the atomic and quantified parts.
- Second, replace the underlined parts with sentence letters. Only use repeat letters for identical parts.

TRUTH-FUNCTIONAL FORM

- Remember: don't look inside quantified sentences.
- $\underline{\forall x\ (Cube(x) \rightarrow Medium(x))}$ P
- $\underline{\forall x\ Cube(x)} \rightarrow \underline{\forall x\ Medium(x)}$ P → Q
- $\underline{Cube(b)} \rightarrow \underline{\exists x\ Cube(x)}$ P → Q
- $\underline{\forall x\ Cube(x)} \rightarrow (\neg\underline{\forall x\ Cube(x)} \rightarrow \underline{\forall x\ \neg Cube(x)})$
  P → (¬P → Q)

Truth-Functional Form

- This results in the truth-functional form of the argument.
- This shows whether an argument is valid in virtue of the connectives.
- Example:

$\forall x$ Cube(x) → $\exists x$ Medium(x)

$\forall x$ Cube(x)

$\exists x$ Medium(x)

P → Q

P

Q

Universal Elimination

- For any variable x, any wff P(x), and any constant c, from ∀x P(x) we can infer P(c).
- Note: the constant c could even have been used in the proof already.

1. ∀x P(x)	
2. P(c)	∀ Elim: 1

Simple Proof

1. All men are mortal
2. Socrates is a man

3. Socrates is mortal

1. $\forall x(Ma(x) \rightarrow Mo(x))$
2. $Ma(s)$

3. $Mo(s)$

1. $\forall x(Ma(x) \rightarrow Mo(x))$
2. $Ma(s)$

3. $Ma(s) \rightarrow Mo(s)$ — $\forall$Elim 1
4. $Mo(s)$ — $\rightarrow$Elim 2,3

Universal Introduction

- For a constant c naming an arbitrary object, any variable x, and any wff P(x), if we show in a subproof that P(c), we can conclude that ∀x P(x).
- Note: the constant c must be new. The step will only work if c only occurs within the subproof.

```
|  | 1. [c]
|  |------
|  | ...
|  | j. P(c)
|
| k. ∀x P(x)          ∀ Intro: 1-j
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Universal Quantifier Proofs

1. $\forall x(P(x) \rightarrow Q(x))$
2. $\forall x(Q(x) \rightarrow R(x))$

   3. $\boxed{a}$

      4. $P(a)$
      5. $P(a) \rightarrow Q(a)$ — $\forall$ Elim 1
      6. $Q(a)$ — $\rightarrow$ Elim 4,5
      7. $Q(a) \rightarrow R(a)$ — $\forall$ Elim 2

      $R(a)$

   $P(a) \rightarrow R(a)$ — $\rightarrow$ Intro

$\forall x(P(x) \rightarrow R(x))$ — $\forall$ Intro

UNIVERSAL QUANTIFIER PROOFS

1. ∀x(P(x) → Q(x))
2. ∀x(Q(x) → R(x))

3. [a]

4. P(a)

5. P(a) → Q(a) ∀ Elim 1

6. Q(a) → Elim 4,5

7. Q(a) → R(a) ∀ Elim 2

8. R(a) → Elim 6,7

9. P(a) → R(a) → Intro 4-8

∀x(P(x) → R(x)) ∀ Intro

'a' is totally arbitrary. We could have gotten this with any letter. e.g. P(j)→R(j)

Universal Quantifier Proofs

1. $\forall x(P(x) \rightarrow Q(x))$
2. $\forall x(Q(x) \rightarrow R(x))$
    3. a
        4. $P(a)$
        5. $P(a) \rightarrow Q(a)$ ∀ Elim 1
        6. $Q(a)$ → Elim 4,5
        7. $Q(a) \rightarrow R(a)$ ∀ Elim 2
        8. $R(a)$ → Elim 6,7
    9. $P(a) \rightarrow R(a)$ → Intro 4-8
10. $\forall x(P(x) \rightarrow R(x))$ ∀ Intro 3-9

1. $\forall x\, P(x) \lor \forall x\, Q(x)$

2. a

3. $\forall x\, P(x)$

$P(a) \lor Q(a)$

$\forall x\, Q(x)$

$P(a) \lor Q(a)$

$P(a) \lor Q(a)$ $\lor$ Elim

$\forall x(P(x) \lor Q(x))$ $\forall$ Intro

can't just plug in 'a' for line 1. 1 is not a universal

1. $\forall x\, P(x) \vee \forall x\, Q(x)$

2. a

3. $\forall x\, P(x)$

4. $P(a)$ — $\forall$ Elim 3

5. $P(a) \vee Q(a)$ — $\vee$ Intro 4

6. $\forall x\, Q(x)$

7. $Q(a)$ — $\forall$ Elim 6

8. $P(a) \vee Q(a)$ — $\vee$ Intro 7

9. $P(a) \vee Q(a)$ — $\vee$ Elim 1,3-5,6-8

10. $\forall x(P(x) \vee Q(x))$ — $\forall$ Intro 2-9

Existential Introduction

- For any variable x, any wff P(x) and any constant c, if we show that P(c), we can conclude that ∃x P(x).
- Note: the constant c could even have been used in the proof already.

1. P(c)	
2. ∃x P(x)	∃ Intro: 1

Existential Elimination

- Existential elimination is like proof by cases, but with only one case representing an infinite number of cases.
- For a constant c naming an arbitrary object, any variable x, and any wff P(x), if we know that ∃x P(x), and we show in a subproof that Q (which does not contain 'c') follows from P(c), we can conclude that Q must be true (outside the subproof).
- Note: the constant c must be new. The step will only work if c only occurs within the subproof.

Existential Elimination

1. ∃x P(x)

   2. [c] P(c)

   ...

   j. Q

7. Q   ∃ Elim: 1,2-j