Puzzle

A, B, and C are each either knights or knaves.

A says “At least one of the three of us is a knight”
B says “At least one of the three of us is a knave”
C says “Some knaves aren’t werewolves”

What can you infer about A, B, and C?

QUANTIFIERS

Wednesday, 13 October

Sentences in FOL

Cube(a)

a is a cube

True in a world if *a* is a cube in that world

∀xCube(x)

For any x, x is a cube

True in a world if every object in that world is a cube

Sentences in FOL

Cube(a)

a is a cube

True in a world if *a* is a cube in that world

∃xCube(x)

For at least one x, x is a cube

True in a world if at least one object in that world is a cube

Cube(x) - Not true or false - not even a sentence

WELL-FORMED FORMULAS

- Both constants and variables are terms, as are functions applied to terms.
- An atomic well-formed formula (wff) is a predicate followed by the appropriate number of terms.
- If P is a wff, so is ¬P.
  If P and Q are wffs, so is (P ∧ Q).
  If P and Q are wffs, so is (P ∨ Q).
  If P and Q are wffs, so is (P → Q).
  If P and Q are wffs, so is (P ↔ Q).

Well-Formed Formulas

- If P is a wff and *v* is a variable, then ∀*v* P is a wff, and any occurrence of *v* in ∀*v* P is said to be bound.
- If P is a wff and *v* is a variable, then ∃*v* P is a wff, and any occurrence of *v* in ∃*v* P is said to be bound.
- Complex wffs are formed out of atomic wffs according to these rules. (Compare to complex and atomic sentences from propositional logic.)
- Wffs are not ambiguous.

Well-Formed Formulas

wffs	not wffs
$\forall x$ Cube(x)	$\forall$ Cube(b)
Taller(Claire, x)	Taller(x $\wedge$ Claire)
$\forall x \exists y$ Smaller(y, x)	Small(a) $\wedge$ Cube(a) $\vee$ Small(b)

- A variable is <u>bound</u> if it is under the scope of a quantifier; a variable is <u>free</u> if it is not bound.
- A wff is a <u>sentence</u> iff it has no free variables.

Well-Formed Formulas

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wffs with free variables	sentences
$Home(u)$	$\forall u\ Home(u)$
$\exists v(Cube(v) \land Small(u))$	$\exists v(Cube(v) \land Small(v))$
$\forall u Large(u) \land Dodec(u)$	$\forall u \exists v(Large(u) \land Dodec(v))$
$\exists v\ \neg Cube(u)$	$\exists v\ Small(v)$

- A wff with free variables is neither true nor false; a sentence is either true or false in a particular world.
- Parentheses are important to whether a wff is a sentence: $\exists v Cube(v) \land Small(v)$ vs. $\exists v(Cube(v) \land Small(v))$

Satisfaction

- An object satisfies a wff with a free variable such as Cube(x) iff it is a cube; an object satisfies Dodec(y) ∧¬Small(y) iff it is a dodecahedron and not small, etc.
- Remember that a free variable is a placeholder. Suppose S(x) is a wff with x as its only free variable.

  An object satisfies S(x) iff the sentence S(b) is true, where b is a constant that names the object.

Satisfaction

- Unnamed objects can also satisfy wffs. An object satisfies S(x) iff the sentence $S(n_1)$ is true, where n_1 is a constant that names the object, possibly temporarily.

- We can use satisfaction to define truth values for sentences containing quantifiers:

- A sentence of the form $\forall x\, S(x)$ is true iff the wff S(x) is satisfied by every object in the domain of discourse.

- A sentence of the form $\exists x\, S(x)$ is true iff the wff S(x) is satisfied by some object in the domain of discourse.

Aristotelian Forms

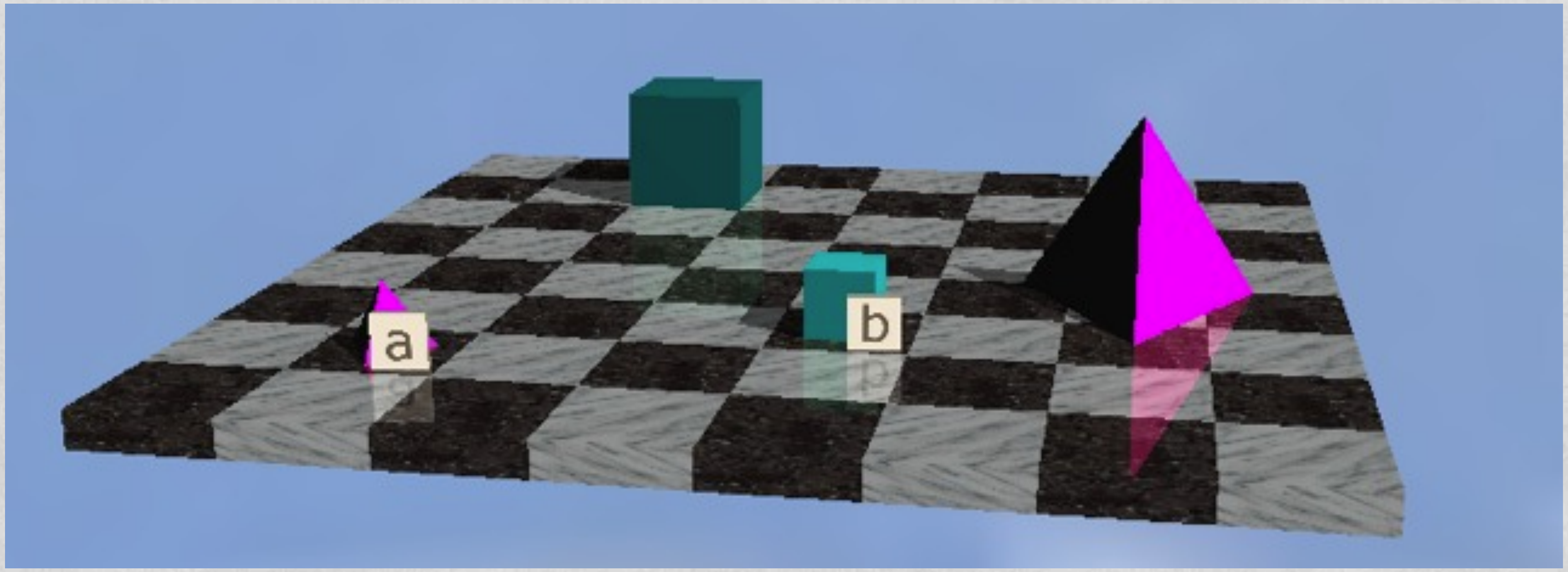
Forms:	Examples:
• All Ps are Qs.	All mammals are animals.
• Some Ps are Qs.	Some mammals live in water.
• No Ps are Qs.	No humans have wings.
• Some Ps are not Qs.	Some birds cannot fly.

Aristotelian Forms

Forms:	QL sentence:
All Ps are Qs.	$\forall x(P(x) \rightarrow Q(x))$
Some Ps are Qs.	$\exists x(P(x) \wedge Q(x))$
No Ps are Qs.	$\forall x(P(x) \rightarrow \neg Q(x))$
Some Ps are not Qs.	$\exists x(P(x) \wedge \neg Q(x))$

SATISFACTION

- $\forall x\ Cube(x)$ F
- $\exists x\ Cube(x)$ T
- $\forall x(Cube(x) \lor Tet(x))$ T
- $\exists x(Cube(x) \lor Dodec(x))$ T
- $\forall x(Cube(x) \to Small(x))$ F
- $\forall x(Cube(x) \to \neg Medium(x))$ F
- $\forall x(Dodec(x) \to Cube(x))$ T
- $\exists x(Cube(x) \to Large(x))$ T



Complex Predicates

Some Ps are Qs	$\exists x(P(x) \land Q(x))$
Some Ps that are also Rs are Qs	$\exists x([P(x) \land R(x)] \land Q(x))$
Some cubes are to the right of *a*	$\exists x(Cubes(x) \land RightOf(x,a))$
Some small cubes are to the right of *a*	$\exists x([Small(x) \land Cube(x)] \land RightOf(x,a))$

Complex Predicates

There is a large cube to the left of *b*

$\exists x(L(x) \wedge C(x) \wedge LO(x,b))$

There is a cube to the left of *b* which is in the same row as *c*

$\exists y(C(y) \wedge LO(y,b) \wedge SR(y,c))$

b is in the same row as a large cube

$\exists x(L(x) \wedge C(x) \wedge SR(b,x))$

Complex Predicates

All Ps are Qs

$\forall x(P(x) \rightarrow Q(x))$

All Ps that are also Rs are Qs

$\forall x([P(x) \wedge R(x)] \rightarrow Q(x))$

All cubes are to the right of *a*

$\forall x(Cubes(x) \rightarrow RightOf(x,a))$

All small cubes are to the right of *a*

$\forall z([Small(z) \wedge Cube(z)] \rightarrow RightOf(z,a))$

Complex Predicates

Every tall boy is a happy painter

$\forall x([T(x) \wedge B(x)] \rightarrow [H(x) \wedge P(x)])$

Not every cube in the same row as *b* is medium

$\neg \forall w([C(w) \wedge SR(w,b)] \rightarrow M(w))$

No cubes in the same row as *b* are medium

$\forall x([C(x) \wedge SR(x,b)] \rightarrow \neg M(x))$

Every cube that is either small or medium is smaller than *b*

$\forall x([C(x) \wedge (S(x) \vee M(x))] \rightarrow Sm(x,b))$

OTHER FORMS

If every block is a cube, then none are dodecs	$\forall x C(x) \rightarrow \forall y \neg D(y)$
Every cube is small if and only if it isn't large	$\forall x(C(x) \rightarrow (S(x) \leftrightarrow \neg L(x)))$
Every cube is either small or medium	$\forall x(C(x) \rightarrow (S(x) \lor M(x)))$
Either every cube is small or every cube is medium	$\forall x(C(x) \rightarrow S(x)) \lor \forall x(C(x) \rightarrow M(x))$

SATISFACTION - AGAIN

$\forall x(x{=}a \rightarrow Tet(x))$ **T**

$\exists x(x{\neq}a \wedge Small(x) \wedge Tet(x))$ **F**

$\forall x((Small(x) \wedge Cube(x)) \rightarrow RightOf(x,a))$ **T**

$\forall x\ RightOf(x,a)$ **F**

$\forall x(Tet(x) \rightarrow (FrontOf(x,b) \rightarrow Small(x)))$ **T**

$\exists x\ SameSize(x,a) \rightarrow x{=}b$ **Not a sentence**