PUZZLE

Some inhabitants of the island of knights and knaves are werewolves. Werewolves can be either knights or knaves. You know that exactly one of A,B,C is a werewolf.

A says "C is a werewolf"
B says "I am not a werewolf"
C says "At least two of us are knaves"

1) Is the werewolf a knight or a knave?
2) If you had to travel with one at night, who would you take?

PROPOSITIONAL LOGIC

Wednesday, 6 October

Soundness Theorem

SOUNDNESS THEOREM (for $\mathcal{F}_T$):

If $\{P_1, P_2, ... P_n\} \vdash$ (in $\mathcal{F}_T$) C then
$\{P_1, P_2, ... P_n\}$ tf-entails C

Negative Criterion

If $\{P_1, P_2, ... P_n\}$ DOES NOT tf-entail C then
$\{P_1, P_2, ... P_n\} \nvdash$ (in $\mathcal{F}_T$) C

Corollaries

If $\{P_1, P_2, ... P_n\} \vdash$(in $\mathcal{F}_T$) C then
$\{P_1, P_2, ... P_n\}$ tf-entails C

If $\{P_1, P_2, ... P_n\}$ DOES NOT tf-entail C then
$\{P_1, P_2, ... P_n\} \nvdash$(in $\mathcal{F}_T$) C

If {} $\vdash$(in $\mathcal{F}_T$) C [$=_{def}$ C is a theorem of $\mathcal{F}_T$] then
{} tf-entails C [= C is a tautology]

All satisfiable sets are consistent or contrapositively
All inconsistent sets are unsatisfiable

Soundness of a whole system

- You can show that none of ∧E ∧I ∨E ∨I →E →I ↔E ↔I ⊥E ⊥I ¬E ¬I reit or making an assumption can introduce the first invalid step so there can't be any invalid steps anywhere in any proof (that uses just these steps).
- So the last line of the proof is a valid step so the conclusion really does follow from the premises on the assumption that there is a legal proof.
- So we say that the system, $\mathcal{F}_{\mathrm{T}}$ is sound.

What about other systems?

- We know that the system $\mathcal{F}_{\mathrm{T}}$ is sound.
- What if we weren't allowed to us the ¬Intro rule? Obviously the resulting system would still be sound. You could still prove only valid arguments. You can just prove less of them.
- But what if we allowed ourselves other rules - like DeMorgan's Laws. Would the system still be sound?

$\mathcal{F}_{\text{T+DeM}}$

- Call $\mathcal{F}_{\text{T+DeM}}$ the system that results from allowing any rules in $\mathcal{F}_{\text{T}}$ and also allows the following rule:

1.	¬(P∨Q)	
2.	¬P∧¬Q	DeM 1

- Is $\mathcal{F}_{\text{T+DeM}}$ sound?
- Answer: YES
- Anything proved in $\mathcal{F}_{\text{T+DeM}}$ really is a valid argument

$\mathcal{F}_{\text{T+DeM}}$

- One way to show soundness is to show that you can't prove anything new - anything provable in $\mathcal{F}_{\text{T+DeM}}$ is also provable in $\mathcal{F}_{\text{T}}$ (but perhaps the proof is longer).
- But we could also directly proof the soundness of the rule: Assuming that $A_1, A_2, \ldots A_k$ really does entail $\neg(P \vee Q)$, then $A_1, A_2, \ldots A_k$ (plus possibly more) really does entail $\neg P \wedge \neg Q$.
- So DeM can't introduce the FIRST invalid step.

$\mathcal{F}_{\text{T+XOR}}$

1. $P \vee Q$
2. P

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3. $\neg Q$ xor 1,2

- Is $\mathcal{F}_{\text{T+xor}}$ sound?
- Answer: NO
- xor CAN introduce the first invalid step

For example, take the proof above. Make P:T Q:T - now steps 1, 2 are valid (since they depend on themselves - the given assumptions) and step 3 is invalid.

BAD RULES ARE <u>REALLY</u> BAD

- If we had xor as a rule (plus the others) our system would be so terrible that it could prove anything at all.

- Example - feel like proving P?

1. ¬P	
2. ¬P∨¬P	∨Intro 1
3. ¬¬P	xor 1,2
4. ⊥	⊥ intro 1,3
5. P	¬Intro 1-4

Which rules would be okay?

- If a rule represents a valid argument (one you could prove anyway by the other rules) then it is okay.
- If a rule represents an invalid argument, or improperly messes with subproofs (reaching into a closed subproof, ending two subproofs at the same time, etc.) it is a bad rule.
- DeM, NegCon, DisjSyll, Modus Tollens, etc. all would be okay rules. Affirming the consequent? Terrible.

Completeness Theorem

- As a matter of fact, the converse of soundness is true - if an argument is tf-valid, then you can do a proof in $\mathcal{F}_{\mathrm{T}}$.
- This is much harder to prove [take 3310 or read chapter 17]. But you can just assume it is true.
- Since $\mathcal{F}_{\mathrm{T}}$ is sound and complete, you can prove all and only the tf-valid arguments. Many other systems of natural deduction have this same quality.

Truth-Functional Completeness

- Is it possible to have a truth-functional sentence that we can't express with our connectives?
- A set of connectives is truth-functionally complete if they allow us to express any truth function.
- We can express exactly one of A+B, neither A nor B, not both A+B, etc. What about 'either 2 or 5 of these 7 variables are true'?
- YES. We can express ANY truth function of arbitrary size or complexity .

Truth-Functional Completeness

P	Q	R	IAmTrueHere
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

Want a sentence true
in exactly these cases?

How about:
(P∧Q∧R)∨(¬P∧Q∧R)∨(¬P∧Q∧¬R)

If a sentence’s truth is completely determined by the truth of its subsentences, then it is equivalent to a sentence like the above using just ¬, ∧, and ∨

Truth-Functional Completeness

- A set of connectives is truth-functionally complete if they allow us to express any truth function.
- Theorem (in book): The set of Boolean Connectives {¬, ∧, and ∨} is truth-functionally complete.
- {¬ and ∨}, {¬ and ∧}, {¬ and →}, {⊥ and →}, are also truth-functionally complete. Some combos, like {¬ and ↔} are not complete (you can't express 'A and B' with only ¬ and ↔).
- Awesome fact: "NAND" [↑] and "NOR" [↓] each by themselves are complete.

Normal Forms

- For various reasons (like automated proof - or proofs of metatheorems like completeness) it is often useful to turn sentences into specific forms.
- The book mentions three kinds - Negated Normal Form (1st step...) Conjunctive Normal Form, Disjunctive Normal Form

Negation Normal Form

- A sentence is in <u>negation normal form (NNF)</u> when any ¬ applies to an atomic sentence and all literals are joined by ∧ or ∨ (and parentheses).
- Any sentence can be put into NNF by getting rid of →s and ↔s and then using double negation and DeMorgan's Laws if necessary.

Conjunctive Normal Form

- A sentence is in conjunctive normal form (CNF) iff it is a conjunction of one or more disjunctions of literals.
- Any sentence in NNF can be put into CNF using the distribution rules.
- Distribution of ∨ over ∧:
  $A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$
- $(P \wedge Q) \vee (R \wedge S) \Leftrightarrow [(P \wedge Q) \vee R] \wedge [(P \wedge Q) \vee S] \Leftrightarrow$

  $[(P \vee R) \wedge (Q \vee R)] \wedge [(P \vee S) \wedge (Q \vee S)]$

Disjunctive Normal Form

- A sentence is in <u>disjunctive normal form (DNF)</u> iff it is a disjunction of one or more conjunctions of literals.
- Any sentence in NNF can be put into DNF using the distribution of $\wedge$ over $\vee$.
- Distribution of $\wedge$ over $\vee$:
  $A \wedge (B \vee C) \;\Leftrightarrow\; (A \wedge B) \vee (A \wedge C)$
- $(P\vee Q)\wedge(R\vee S) \Leftrightarrow [(P\vee Q)\wedge R]\vee[(P\vee Q)\wedge S] \Leftrightarrow$
  $[(P\wedge R)\vee(Q\wedge R)] \vee [(P\wedge S)\vee(Q\wedge S)]$

LIMITS OF TRUTH-FUNCTIONS

a is a cube
b is not a cube

$a \neq b$

This is provable if you add the identity rules

a is a cube
b is not a cube

There are at least two things

This is still not

LIMITS OF TRUTH-FUNCTIONS

All men are mortal
Socrates is a man

Socrates is mortal

All men are tall
Not every man is bald

Some tall people aren't bald

No apples are rotten
Some fruits are rotten

Some fruits aren't apples

For any number, there is a larger prime number

There is no largest prime number

None are truth-functionally valid

- We need a stronger logical system

QUANTIFIERS

Two quantifier symbols:

- $\forall$ means "everything" or "for all".
- $\exists$ means "something" or "there exists at least one".
- Just these two quantifiers can be used to capture many of the quantifications we want to talk about. For example, all, every, any, none, not all of, some, some are not, at least one, at least two, exactly two, etc.

Example Sentences

- $\forall x\ Cube(x)$ - Everything is a cube
- $\exists x\ Cube(x)$ - Something is a cube
- $\forall x(Cube(x) \land Small(x))$ - Everything is a small cube
- $\exists x(Cube(x) \land Small(x))$ - Something is a small cube
- $\forall x(Cube(x) \rightarrow Small(x))$ - Every cube is small
- $\forall x(Tet(x) \rightarrow Cube(x))$ - Every tet is a cube
- $\neg \exists x(Cube(x) \land Large(x))$ - There aren't any large cubes