Puzzle

Once I visited the island of knights and knaves, and I met A and B. I asked A "Is either of you a knight?" He responded and after thinking about it, I knew the answer to my question.

What are A and B?

The Soundness of $\mathcal{F}_{\mathrm{T}}$

Friday, 1 October

Soundness Theorem

SOUNDNESS THEOREM (for $\mathcal{F}_T$):

If $\{P_1, P_2, ... P_n\} \vdash$ (in $\mathcal{F}_T$) C then
$\{P_1, P_2, ... P_n\}$ tf-entails C

Negative Criterion

If $\{P_1, P_2, ... P_n\}$ DOES NOT tf-entail C then
$\{P_1, P_2, ... P_n\} \nvdash$ (in $\mathcal{F}_T$) C

Soundness Theorem

- For some formal proof system to be sound, it means that anything you can prove in that system really is a valid argument.

- P→Q, Q therefore P really is invalid, but how can I be so sure that I can't prove this in $\mathcal{F}_T$? What if I were really clever?

- I need to show that some relevant fact is true about every one of the infinite number of possible proofs in $\mathcal{F}_T$. Obviously, "check them all" is not the answer.

SOUNDNESS THEOREM

SOUNDNESS THEOREM (for $\mathcal{F}_T$):

If $\{P_1, P_2, ... P_n\} \vdash$(in $\mathcal{F}_T$) C then
$\{P_1, P_2, ... P_n\}$ tf-entails C

[Here I will follow the sketch in section 8.3 in the book]

This is a conditional. I will assume the antecedent (we can do a proof) and try to prove the consequent (the conclusion really does follow from the premises). One way to prove this is to prove the apparently stronger claim that of every step in every line of every proof, the sentence on that line is a consequence of the assumptions "in force" at that line. If that is true of every line, it is true of the last line and so the conclusion would follow from the premises since they are the only assumptions in force.

Soundness Theorem

- Call a line where the sentence on the line doesn't follow from the assumptions in force on that line an "invalid step". We will prove that no lines in any proof are invalid steps by showing that there can't be a first invalid step.
- We will show no first invalid step by showing that none of our rules could justify the first invalid step.
- We will show this one rule at a time - $\wedge$Elim can't introduce the first invalid step, $\leftrightarrow$Intro can't introduce the first invalid step, etc.

$\wedge$ELIM

- To use $\wedge$Elim, we have $P \wedge Q$ on some line with assumptions $A_1, A_2, \ldots A_k$ in force. Then later, we infer P on some line with all of $A_1, A_2, \ldots A_k$ still in force (plus possibly more).
- So assuming the earlier $P \wedge Q$ step was a valid step, $P \wedge Q$ really does follow from $A_1, A_2, \ldots A_k$
- But this means that P really does follow from $A_1, A_2, \ldots A_k$ (plus possibly more). So P can't be the FIRST invalid step.

∨ELIM

- To use ∨Elim, we have P∨Q on some line with assumptions $A_1, A_2, \ldots A_k$ in force. Then later, we infer R on some line with all of $A_1, A_2, \ldots A_k$ still in force, plus P in force (plus set X). Then later we infer R on some line with all of $A_1, A_2, \ldots A_k$ still in force, plus Q in force (plus set X). Now we can infer R based on A_1, $A_2, \ldots A_k$ plus X.

- So assuming the earlier steps were valid, P∨Q really does follow from $A_1, A_2, \ldots A_k$, R follows from $A_1, A_2, \ldots A_k$, plus P plus X and R follows from $A_1, A_2, \ldots A_k$ plus Q plus X.

∨ELIM

So we know $A_1, A_2, \ldots A_k$ entails P∨Q

we know $A_1, A_2, \ldots A_k$, X, P entails R

we know $A_1, A_2, \ldots A_k$, X, Q entails R

We need to show that $A_1, A_2, \ldots A_k$, X entails R

Assume this is false. Then it is possible for $A_1, A_2, \ldots A_k$, X all true and R false

But this very assignment must make $A_1, A_2, \ldots A_k$, X all true and so must make P∨Q true as well.

But it can't make P true (while still making R false) and it can't make Q true (while still making R false). So there can't be any such assignment. So assuming the earlier lines were all valid steps, ∨Elim is a valid step too.

Soundness of a whole system

- You can show that none of ∧E ∧I ∨E ∨I →E →I ↔E ↔I ⊥E ⊥I ¬E ¬I reit or making an assumption can introduce the first invalid step so there can't be any invalid steps anywhere in any proof (that uses just these steps).
- So the last line of the proof is a valid step so the conclusion really does follow from the premises on the assumption that there is a legal proof.
- So we say that the system, $\mathcal{F}_{\mathrm{T}}$ is sound.

Corollaries

If $\{P_1, P_2, ... P_n\} \vdash$(in $\mathcal{F}_T$) C then
{$P_1, P_2, ... P_n$} tf-entails C

If {$P_1, P_2, ... P_n$} DOES NOT tf-entail C then
$\{P_1, P_2, ... P_n\} \nvdash$(in $\mathcal{F}_T$) C

If $\{\} \vdash$(in $\mathcal{F}_T$) C [$=_{def}$ C is a theorem of $\mathcal{F}_T$] then
{} tf-entails C [= C is a tautology]

All satisfiable sets are consistent or contrapositively
All inconsistent sets are unsatisfiable

What about other systems?

- We know that the system $\mathcal{F}_{\mathrm{T}}$ is sound.
- What if we weren't allowed to us the ¬Intro rule? Obviously the resulting system would still be sound. You still can only prove valid arguments. You can just prove less of them.
- But what if we allowed ourselves other rules - like DeMorgan's Laws. Would the system still be sound?

$\mathcal{F}_{\text{T+DeM}}$

- Call $\mathcal{F}_{\text{T+DeM}}$ the system that results from allowing any rules in $\mathcal{F}_{\text{T}}$ and also allows the following rule:

1. ¬(P∨Q)

2. ¬P∧¬Q   DeM 1

- Is $\mathcal{F}_{\text{T+DeM}}$ sound?
- Answer: YES
- Anything proved in $\mathcal{F}_{\text{T+DeM}}$ really is a valid argument

$\mathcal{F}_{\mathrm{T+DeM}}$

- One way to show soundness is to show that you can't prove anything new - anything provable in $\mathcal{F}_{\mathrm{T+DeM}}$ is also provable in $\mathcal{F}_{\mathrm{T}}$ (but perhaps the proof is longer).
- But we could also directly proof the soundness of the rule: Assuming that $A_1, A_2, \ldots A_k$ really does entail $\neg(P \vee Q)$, then $A_1, A_2, \ldots A_k$ (plus possibly more) really does entail $\neg P \wedge \neg Q$.
- So DeM can't introduce the FIRST invalid step.

$\mathcal{F}_{\mathrm{T+XOR}}$

1. P∨Q
2. P
3. ¬Q xor 1,2

- Is $\mathcal{F}_{\mathrm{T+xor}}$ sound?
- Answer: NO
- xor CAN introduce the first invalid step

For example, take the proof above. Make P:T Q:T - now steps 1, 2 are valid (since they depend on themselves - the given assumptions) and step 3 is invalid.

BAD RULES ARE REALLY BAD

- If we had xor as a rule (plus the others) our system would be so terrible that it could prove anything at all.
- Example - feel like proving P?

1. $\neg P$	
2. $\neg P \vee \neg P$	$\vee$Intro 1
3. $\neg\neg P$	xor 1,2
4. $\bot$	$\bot$ intro 1,3
5. P	$\neg$Intro 1-4

Which rules would be okay?

- If a rule represents a valid argument (one you could prove anyway by the other rules) then it is okay.
- If a rule represents an invalid argument, or improperly messes with subproofs (reaching into a closed subproof, ending two subproofs at the same time, etc.) it is a bad rule.
- DeM, NegCon, DisjSyll, Modus Tollens, etc. all would be okay rules. Affirming the consequent? Terrible.

Completeness Theorem

- As a matter of fact, the converse of soundness is true - if an argument is tf-valid, then you can do a proof in $\mathcal{F}_{\mathrm{T}}$.
- This is much harder to prove [take 3310 or read chapter 17]. But you can just assume it is true.
- Since $\mathcal{F}_{\mathrm{T}}$ is sound and complete, you can prove all and only the tf-valid arguments. Many other systems of natural deduction have this same quality.