Puzzle

On the island of knights and knaves, I meet A, B, and C and hear A make a muffled sound, but I couldn't make out the words. I asked B "What did A say?" B says, "A said there is one knight among us." C says "B is lying."

What are B and C?

TESTING VALIDITY II

Wednesday, 29 September

The Short Table Method

- To show that a conclusion is a tautological consequence of the premises, producing a proof in $\mathcal{F}_{\mathrm{T}}$ suffices.
- To show that a conclusion is not a tautological consequence of the premises, a truth value assignment (TVA) that makes all of the premises true and the conclusion false at the same time suffices.
- One way of detecting consequence is to assume it is not a consequence and then produce such a row.

Inconsistency and Logical Consequence

- Notice in all these cases, we are trying to get an assignment where all the premises are true and the conclusion is false. If this is impossible, then the argument is valid.

- $\{P_1, P_2, ... P_n, \neg C\}$ is inconsistent iff
  $\{P_1, P_2, ... P_n\}$ logically entails C

- $\{P_1, P_2, ... P_n\}$ tautologically entails C iff
  $\{P_1, P_2, ... P_n\} \vdash (\text{in } \mathcal{F}_T)$ C

(Soundness and Completeness Theorems)

Inconsistency and Logical Consequence

$\{P_1, P_2\}$ logically entails C

$=_{def}$ No way for P_1 and P_2 to be true and C false

Thus no way for P_1 to be true and $P_2 \rightarrow C$ false

$=_{def}$ $\{P_1\}$ logically entails $P_2 \rightarrow C$

Thus $\{P_1\}$ logically entails $\neg C \rightarrow \neg P_2$

Thus $\{P_1, \neg C\}$ logically entails $\neg P_2$

$=_{def}$ No way for P_1 and $\neg C$ to be true and $\neg P_2$ false

Thus No way for P_1 and $\neg C$ to be true and P_2 true

$=_{def}$ $\{P_1, \neg C, P_2\}$ is inconsistent

Provability in a Formal System

- If it is possible to prove C from $\{P_1, P_2, ... P_n\}$ using just the truth functional rules we say that:

$$\{P_1, P_2, ... P_n\} \vdash (\text{in } \mathcal{F}_T)\ C$$

Relationships Between Proofs

Assume $\{P_1, P_2\} \vdash C$

Thus $\{P_1\} \vdash P_2 \rightarrow C$ (think →Intro)

Thus $\{P_1\} \vdash \neg C \rightarrow \neg P_2$ (think contraposition)

Thus $\{P_1, \neg C\} \vdash \neg P_2$ (think →Intro)

Thus $\{P_1, \neg C\} \vdash P_2 \rightarrow \bot$ (think →Intro)

Thus $\{P_1, \neg C, P_2\} \vdash \bot$ (think →Intro)

$=_{def}$ $\{P_1, \neg C, P_2\}$ is [proof theoretically] inconsistent

Relationships Between Proofs

Assume $\{P_1, P_2\} \vdash C$

Thus $\{P_1\} \vdash P_2 \rightarrow C$ (think $\rightarrow$Intro)

Thus $\{P_1\} \vdash \neg C \rightarrow \neg P_2$ (think contraposition)

Thus $\{P_1, \neg C\} \vdash \neg P_2$ (think $\rightarrow$Intro)

Thus $\{P_1, \neg C\} \vdash P_2 \rightarrow \bot$ (think $\rightarrow$Intro)

Thus $\{P_1, P_2, \neg C\} \vdash \bot$ Reductio ($\neg$Intro)

$=_{def}$ $\{P_1, P_2, \neg C\}$ is [proof theoretically] inconsistent

Connecting Proofs to Truth

- If we think of subproofs as conditionals, all of the rules represent valid arguments.
- Therefore each of the rules we use is Truth-Preserving. If the assumptions we make are true, then each new line would be true as well.

SOUNDNESS THEOREM (for $\mathcal{F}_{\mathrm{T}}$):

If $\{P_1, P_2, \ldots. P_n\} \vdash$ (in $\mathcal{F}_{\mathrm{T}}$) C then
$\{P_1, P_2, \ldots. P_n\}$ tf-entails C

SOUNDNESS THEOREM

SOUNDNESS THEOREM (for $\mathcal{F}_T$):

If $\{P_1, P_2, ... P_n\} \vdash$(in $\mathcal{F}_T$) C then
$\{P_1, P_2, ... P_n\}$ tf-entails C

Think Contrapositively

If $\{P_1, P_2, ... P_n\}$ DOES NOT tf-entail C then
$\{P_1, P_2, ... P_n\} \nvdash$(in $\mathcal{F}_T$) C

Therefore a falsifying assignment shows that you can't do a proof

Short Table Method

$A \rightarrow B$

$A \vee C$

$B \vee D$

Tautologically Valid or not?

If not valid, some row of the truth table looks like this:

A	B	C	D	$A \rightarrow B$	$A \vee C$	$B \vee D$
				T	T	F

Short Table Method

A	B	C	D	$A \rightarrow B$	$A \vee C$	$B \vee D$
F	F	T	F	T	T	F

Since $B \vee D$ false, B: F and D: F

Since $A \rightarrow B$ true and $\neg B$, A: F

Since $A \vee C$ true and $\neg A$, C:T

Since this row is correct, the argument is invalid

Short Table Method

P	Q	R	S	¬P	→	Q	(R	∧	S)	∨	¬R	Q	→	¬R
T/F	T	T	T		T			T		T	F		F	F

Since Q→¬R false, Q:T and ¬R: F so R:T

Since (R∧S)∨¬R true and R, R∧S:T so R:T and S:T

Since ¬P→Q and Q, we know what about ¬P?

It doesn't matter what P is

Since this row is correct, the argument is invalid

Short Table Method

P	R	S	$\neg P \rightarrow (R \wedge S)$	$S \leftrightarrow P$	$S \wedge P$
			T	T	F

No obvious way to start this - so make a guess. If it works, great. If not, make sure to check the other possibility!

Short Table Method

P	R	S	$\neg P \rightarrow (R \wedge S)$	$S \leftrightarrow P$	$S \wedge P$
F		F	T T	T	F

Since $S \wedge P$ false, either S false or P false. Assume S false.

Since $S \leftrightarrow P$ and $\neg S$, P: F

Since $\neg P \rightarrow (R \wedge S)$ and $\neg P$, $R \wedge S$:T so R:T and S:T

Nope - we already made S false. So our assumption (S false) can't lead to a counterexample.

NOW BACK UP!!

SHORT TABLE METHOD

P	R	S	$\neg P \rightarrow (R \wedge S)$	$S \leftrightarrow P$	$S \wedge P$
F		T	T	T	F

Since $S \wedge P$ false, either S false or P false.

Since S: F can't lead to a counterexample, if there is one, it has S:T - This means P false.

But S:T and P: F means that premise 2 is false.

Since there can't be a counterexample,
this argument is valid