Lady or the Tiger?
(puzzle book by Raymund Smullyan)

There are two doors. Behind each is a lady or a tiger (not both). Each door has a statement on it. If a lady is in room 1, that door's statement is true, otherwise it is false. If a lady is in room 2, that door's statement is false, otherwise it is true.

Door 1

It makes no difference which door you pick

Door 2

There is a lady in the other room

Testing Validity

Monday, 27 September

Pushing Negations Inside

DeMorgan's Laws

$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$

$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$

Negated Conditional

$\neg(P \rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$

Negated Biconditional

$\neg(P \leftrightarrow Q) \Leftrightarrow (\neg P \leftrightarrow Q)$

With repeated applications of these rules, we can convert any sentence with main connective ¬ into something with a different main connective.

Or get rid of any particular connectives that we don't like

Problem using Taut Con

$S \vee (P \leftrightarrow Q)$

$S \rightarrow R$

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$
2. $S \rightarrow R$

$P \vee (Q \rightarrow R)$

1.	$S \lor (P \leftrightarrow Q)$	
2.	$S \to R$	
3.	$\neg(P \lor (Q \to R))$	for ¬I
4.	$\neg P \land \neg(Q \to R)$	DeMorgans 3
5.	$\neg P$	∧Elim 4
6.	$\neg(Q \to R)$	∧Elim 4
7.	$Q \land \neg R$	NegCon 6
8.	Q	∧Elim 7
9.	$\neg R$	∧Elim 7
10.	$\neg S$	Modus Tollens 2,9
11.	$P \leftrightarrow Q$	Disjunctive Syllogism 1,10
12.	P	↔Elim 8,11
13.	$\bot$	⊥ Intro 5,12
	$P \lor (Q \to R)$	

1. $S \vee (P \leftrightarrow Q)$		
2. $S \rightarrow R$		
3. $\neg(P \vee (Q \rightarrow R))$		for ¬I
4. $\neg P \wedge \neg(Q \rightarrow R)$	Taut Con 3	DeMorgans 3
5. $\neg P$	∧Elim 4	
6. $\neg(Q \rightarrow R)$	∧Elim 4	
7. $Q \wedge \neg R$	Taut Con 6	NegCon 6
8. Q	∧Elim 7	
9. $\neg R$	∧Elim 7	
10. $\neg S$	Taut Con 2,9	Modus Tollens 2,9
11. $P \leftrightarrow Q$	Taut Con 1,10	Disjunctive Syllogism 1,10
12. P	↔Elim 11,12	
13. $\bot$	⊥ Intro 5,12	
14. $P \vee (Q \rightarrow R)$	¬Intro 3-13	

HARDER PROOFS

| $P \leftrightarrow (Q \leftrightarrow R)$
|---
| $(P \leftrightarrow Q) \leftrightarrow R$

This is by no means trivial!

(like it is with $\wedge$ and $\vee$)

$P \leftrightarrow (Q \leftrightarrow R)$ does NOT mean

$P \Leftrightarrow Q \Leftrightarrow R$

For example, $P \not\Leftrightarrow P \leftrightarrow P$

$P \leftrightarrow (P \leftrightarrow P)$ is NOT a tautology

HARDER PROOFS

$P \leftrightarrow (Q \leftrightarrow R)$

$(P \leftrightarrow Q) \leftrightarrow R$

$(P \leftrightarrow Q) \vee (P \leftrightarrow R) \vee (Q \leftrightarrow R)$

Premise and conclusion are each true iff one or three of P,Q,R are true

True because there are only two truth values

Invalid Arguments

$(S \wedge P) \leftrightarrow Q$	1. $(S \wedge P) \leftrightarrow Q$	
$T \wedge S$	2. $T \wedge S$	
$(T \rightarrow P) \rightarrow \neg Q$	3. $T \rightarrow P$	for $\rightarrow$Intro
	4. T	$\wedge$Elim 2
	5. S	$\wedge$Elim 2
	6. P	$\rightarrow$Elim 3,4
	7. $S \wedge P$	$\wedge$Intro 5,6
	8. Q	$\leftrightarrow$Elim 1,7
How could you get $\neg Q$?? →	$\neg Q$	
	$(T \rightarrow P) \rightarrow \neg Q$	

Invalid Arguments

1. $(S \wedge P) \leftrightarrow Q$
2. $T \wedge S$

3. $T \rightarrow P$

4. T
5. S
6. P
7. $S \wedge P$
8. Q

$\neg Q$

$(T \rightarrow P) \rightarrow \neg Q$

A counterexample makes all of the premises true and the conclusion false.

T, S, P, and Q all true makes all the premises true

and the conclusion false

The Hard Way

=1=	=2=	=3=	=4=	(1)		(2)	(3)		
S	P	Q	T	(S∧P)	↔ Q	T∧S	(T →	P) →	¬Q
T	T	T	T	T	T	T	T	F	F
T	T	T	F	T	T	F	T	F	F
T	T	F	T	T	F	T	T	T	T
T	T	F	F	T	F	F	T	T	T
T	F	T	T	F	F	T	F	T	F
T	F	T	F	F	F	F	T	F	F
T	F	F	T	F	T	T	F	T	T
T	F	F	F	F	T	F	T	T	T
F	T	T	T	F	F	F	T	F	F
F	T	T	F	F	F	F	T	F	F
F	T	F	T	F	T	F	T	T	T
F	T	F	F	F	T	F	T	T	T
F	F	T	T	F	F	F	F	T	F
F	F	T	F	F	F	F	T	F	F
F	F	F	T	F	T	F	F	T	T
F	F	F	F	F	T	F	T	T	T

Notice all premises true, conclusion false

The Short Table Method

- To show that a conclusion is a tautological consequence of the premises, producing a proof in $\mathcal{F}_{\mathrm{T}}$ suffices.
- To show that a conclusion is not a tautological consequence of the premises, a truth value assignment (TVA) that makes all of the premises true and the conclusion false at the same time suffices.
- One way of detecting consequence is to assume it is not a consequence and then produce such a row.

The Short Table Method

$S \rightarrow \neg P$

$\neg Q \vee U$

$(P \wedge Q) \rightarrow R$

Tautologically Valid or not?

If there is a counterexample, it must make the premises true and the conclusion false. Therefore it makes $(P \wedge Q) \rightarrow R$ False. So it makes $P \wedge Q$ True and R False. Since it makes $P \wedge Q$ True, it makes P True and Q True. By premise 1, $S \rightarrow \neg P$ is True and since this assignment makes P True, it must make S False. By Premise 2, $\neg Q \vee U$ is True. But since Q is true, this assignment must make U true.

Counterexample: P:T, Q:T, R: F, S: F, U:T

The Short Table Method

A ∧ B

(A∧C)↔D

B∧(D∨¬C)

Tautologically Valid or not?

If there is a counterexample, it must make the premises true and the conclusion false. Therefore it makes A∧B True. So it makes A True and B True. Since it makes B∧(D∨¬C) False, it must make either B False or D∨¬C False. But B is true, so D∨¬C must be False. This means that D is False and C is True. But now we have A and C both True and D False which makes premise 2 False. So there can be no such assignment.

Alleged counterexample must have A:T, B:T, D: F, C: F to get premise 1 True and conc False, but then premise 2 is also false.

Inconsistency and Logical Consequence

- Notice in all these cases, we are trying to get an assignment where all the premises are true and the conclusion is false. If this is impossible, then the argument is valid.
- $\{P_1, P_2, ... P_n, \neg C\}$ is inconsistent iff
  $\{P_1, P_2, ... P_n\}$ logically entails C
- $\{P_1, P_2, ... P_n\}$ tautologically entails C iff
  $\{P_1, P_2, ... P_n\} \vdash (\text{in } \mathcal{F}_T)$ C

(Soundness and Completeness Theorems)

Inconsistency and Logical Consequence

$\{P_1, P_2\}$ logically entails C

$=_{def}$ No way for P_1 and P_2 to be true and C false

Thus no way for P_1 to be true and $P_2 \rightarrow C$ false

$=_{def}$ $\{P_1\}$ logically entails $P_2 \rightarrow C$

Thus $\{P_1\}$ logically entails $\neg C \rightarrow \neg P_2$

Thus $\{P_1, \neg C\}$ logically entails $\neg P_2$

$=_{def}$ No way for P_1 and $\neg C$ to be true and $\neg P_2$ false

Thus No way for P_1 and $\neg C$ to be true and P_2 true

$=_{def}$ $\{P_1, \neg C, P_2\}$ is inconsistent