Valid Arguments?

$\neg G \to \neg (P \to A)$
$\neg P$
$\therefore G$

If God does not exist, then it is not true that if I pray, then my prayers will be answered. I don't pray. Therefore, there is a God.

$(P \to A) \to G$
$\neg P$
$\therefore G$

If it is true that if I pray then my prayers will be answered, then there is a God. But I don't pray. Therefore, there is a God.

PROOFS WITH CONDITIONALS 3

Friday, 24 September

Formal Proof Rules

- ↔ <u>Introduction</u>: from a proof from P to Q and a proof from Q to P, we can infer P ↔ Q.

| 1. P
| ---
| ...
| j. Q

| k. Q
| ---
| ...
| m. P

k. P ↔ Q   ↔ Intro: 1-j, k-m

Paradoxes of Material Implication

Last time we showed:

| $\neg P \vee Q$
|———
| $P \rightarrow Q$

We did this by showing that:

| $\neg P$
|———
| $P \rightarrow Q$

and

| Q
|———
| $P \rightarrow Q$

| 1. $\neg P$
|———
| | 2. P — for $\rightarrow$Intro
| |———
| | 3. $\bot$ — $\bot$ Intro 1,2
| | 4. Q — $\bot$ Elim 3
| 5. $P \rightarrow Q$ — $\rightarrow$Intro 2-4

| 1. Q
|———
| | 2. P — for $\rightarrow$Intro
| |———
| | 3. Q — Reit 1
| 5. $P \rightarrow Q$ — $\rightarrow$Intro 2-3

The Other Direction

Example:

$P \to Q$

$\neg P \vee Q$

1. $P \to Q$
2. $\neg(\neg P \vee Q)$ — for ¬Intro
3. $\neg P$ — for ¬Intro
4. $\neg P \vee Q$ — vIntro 3
5. $\bot$ — ⊥ Intro 2,4
6. P — ¬Intro 3-5
7. Q — →Elim 1,6
8. $\neg P \vee Q$ — vIntro 7

$\bot$ — ⊥ Intro

$\neg P \vee Q$ — ¬Intro 2-

The Other Direction

Example:

$P \rightarrow Q$

$\neg P \lor Q$

1.	$P \rightarrow Q$	
2.	$\neg(\neg P \lor Q)$	for ¬Intro
3.	$\neg P$	for ¬Intro
4.	$\neg P \lor Q$	∨Intro 3
5.	$\bot$	⊥ Intro 2,4
6.	P	¬Intro 3-5
7.	Q	→Elim 1,6
8.	$\neg P \lor Q$	∨Intro 7
9.	$\bot$	⊥ Intro 2,8
10.	$\neg P \lor Q$	¬Intro 2-9

Proving Biconditionals

We have now proved:

$$\begin{array}{|l} P \rightarrow Q \\ \hline \neg P \lor Q \end{array} \quad \text{and} \quad \begin{array}{|l} \neg P \lor Q \\ \hline P \rightarrow Q \end{array}$$

Therefore we could prove:

$$\begin{array}{|l} \\ \hline (P \rightarrow Q) \leftrightarrow (\neg P \lor Q) \end{array}$$

PROVING BICONDITIONALS

(P → Q)↔(¬P ∨ Q)

1. P→Q — for ↔Intro

¬P ∨ Q

¬P ∨ Q — for ↔Intro

P→Q

(P → Q)↔(¬P ∨ Q) — ↔Intro

1. P→Q

¬P ∨ Q

¬P ∨ Q

P→Q

(P → Q)↔(¬P ∨ Q) ↔Intro 1-10, 11-18

1. P → Q
2. ¬(¬P ∨ Q) for ¬Intro
3. ¬P for ¬Intro
4. ¬P ∨ Q ∨Intro 3
5. ⊥ ⊥ Intro 2,4
6. P ¬Intro 3-5
7. Q →Elim 1,6
8. ¬P ∨ Q ∨Intro 7
9. ⊥ ⊥ Intro 2,7
10. ¬P ∨ Q ¬Intro 2-9

1. ¬P ∨ Q
2. P for →Intro
3. ¬P for ∨ Elim
4. ⊥ ⊥ Intro 2,3
5. Q ⊥ Elim 4
6. Q for ∨ Elim
7. Q ∨Elim 1,3-5,6-6
8. P→Q →Intro 2-7

Biconditionals and Equivalence

We have now proved:

$$\vdash (P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$$

If a biconditional is a logical truth then the two parts are logically equivalent:

$$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$

Chains of Equivalence

$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$ ← When a conditional is true

By DeMorgan's

$(\neg P \vee Q) \Leftrightarrow \neg(P \wedge \neg Q)$

Therefore

$(P \rightarrow Q) \Leftrightarrow \neg(P \wedge \neg Q)$

and so

$\neg(P \rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$ ← When a conditional is false

NEGATED CONDITIONALS

$(P \to Q) \leftrightarrow \neg(P \land \neg Q)$

1. $P \to Q$ for $\leftrightarrow$Intro

$\neg(P \land \neg Q)$

$\neg(P \land \neg Q)$ for $\leftrightarrow$Intro

$P \to Q$

$(P \to Q) \leftrightarrow \neg(P \land \neg Q)$ $\leftrightarrow$Intro

1. P → Q — for ↔ Intro

2. P ∧ ¬Q — for ¬ Intro

3. P — ∧Elim2

4. ¬Q — ∧Elim2

5. Q — →Elim 1,3

6. ⊥ — ⊥Intro 4,5

7. ¬(P ∧ ¬Q) — ¬Intro 2-6

8. ¬(P ∧ ¬Q) — for ↔ Intro

9. P — for → Intro

10. ¬Q — for ¬ Intro

11. P ∧ ¬Q — ∧Intro 9,10

12. ⊥ — ⊥Intro 8,11

Q — ¬ Intro

P → Q — →Intro 9-

(P → Q) ↔ ¬(P ∧ ¬Q) — ↔Intro

1. P→Q for ↔Intro

2. P∧¬Q for ¬ Intro

3. P ∧Elim2

4. ¬Q ∧Elim2

5. Q →Elim1,3

6. ⊥ ⊥Intro 4,5

7. ¬(P ∧ ¬Q) ¬Intro 2-6

8. ¬(P ∧ ¬Q) for ↔Intro

9. P for →Intro

10. ¬Q for ¬ Intro

11. P∧¬Q ∧Intro 9,10

12. ⊥ ⊥Intro 8,11

13. Q ¬ Intro 10-12

14. P→Q →Intro 9-13

15. (P → Q) ↔ ¬(P ∧ ¬Q) ↔Intro 1-7, 8-14

Pushing Negations Inside

DeMorgan's Laws

$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$

$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$

Negated Conditional

$\neg(P \rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$

Negated Biconditional

$\neg(P \leftrightarrow Q) \Leftrightarrow (\neg P \leftrightarrow Q)$

With repeated applications of these rules, we can convert any sentence with main connective ¬ into something with a different main connective.

Or get rid of any particular connectives that we don't like

Double Reductios

The Law of the Excluded Middle

$P \vee \neg P$

1. $\neg(P \vee \neg P)$	for $\neg$Intro
2. P	for $\neg$Intro
3. $P \vee \neg P$	$\vee$Intro 2
4. $\bot$	$\bot$Intro 1,3
5. $\neg P$	$\neg$Intro 2-4
6. $P \vee \neg P$	$\vee$Intro 5
$\bot$	
$P \vee \neg P$	$\neg$Intro

Double Reductios

The Law of the Excluded Middle

$\vdash P \vee \neg P$

1. $\neg(P \vee \neg P)$	for ¬Intro
2. P	for ¬Intro
3. $P \vee \neg P$	∨Intro 2
4. $\bot$	⊥Intro 1,3
5. $\neg P$	¬Intro 2-4
6. $P \vee \neg P$	∨Intro 5
7. $\bot$	⊥Intro 1,6
$P \vee \neg P$	¬Intro 1-7

Double Reductios

$A \to B$

$\neg A \to B$

B

1.	$A \to B$	
2.	$\neg A \to B$	
3.	$\neg B$	for $\neg$ Intro
4.	A	for $\neg$ Intro
5.	B	$\to$Elim 1,3
6.	$\bot$	$\bot$Intro 3,5
7.	$\neg A$	$\neg$ Intro 4-6
8.	B	$\to$Elim 2,7
9.	$\bot$	$\bot$Intro 3,8
	B	$\neg$Intro 3-9

Using LEM

1.	A→B	
2.	¬A→B	
3.	¬B	for ¬ Intro
4.	A	for ¬ Intro
5.	B	→Elim 1,3
6.	⊥	⊥Intro 3,5
7.	¬A	¬ Intro 4-6
8.	B	→Elim 2,7
9.	⊥	⊥Intro 3,8
10.	B	¬Intro 3-9

1.	A→B	
2.	¬A→B	
3.	A∨ ¬A	LEM
4.	A	for ∨Elim
5.	B	→Elim 1,3
6.	¬A	for ∨Elim
7.	B	→Elim 2,6
8.	B	∨Elim 3,4-5,6-7

How to Really do Proofs

6.42 in LPL book

$\neg A \vee \neg(\neg B \wedge (\neg A \vee B))$

1. $\neg(\neg A \vee \neg(\neg B \wedge (\neg A \vee B)))$ for $\neg$I
2. $A \wedge (\neg B \wedge (\neg A \vee B))$ DeMorgans
3. A $\wedge$Elim
4. $\neg B$ $\wedge$Elim
5. $\neg A \vee B$ $\wedge$Elim
6. $\perp$ from 3-5

$\neg A \vee \neg(\neg B \wedge (\neg A \vee B))$

Really Hard Proofs

| $P \leftrightarrow (Q \leftrightarrow R)$
|---
| $(P \leftrightarrow Q) \leftrightarrow R$

This is by no means trivial!

(like it is with $\wedge$ and $\vee$)

$P \leftrightarrow (Q \leftrightarrow R)$ does <u>NOT</u> mean

$P \Leftrightarrow Q \Leftrightarrow R$

For example, $P \not\Leftrightarrow P \leftrightarrow P$

$P \leftrightarrow (P \leftrightarrow P)$ is NOT a tautology