

VALID ARGUMENTS?

$$\begin{array}{|l} \neg G \rightarrow \neg(P \rightarrow A) \\ \neg P \\ \hline G \end{array}$$

If God does not exist, then it is not true that if I pray, then my prayers will be answered. I don't pray. Therefore, there is a God.

$$\begin{array}{|l} (P \rightarrow A) \rightarrow G \\ \neg P \\ \hline G \end{array}$$

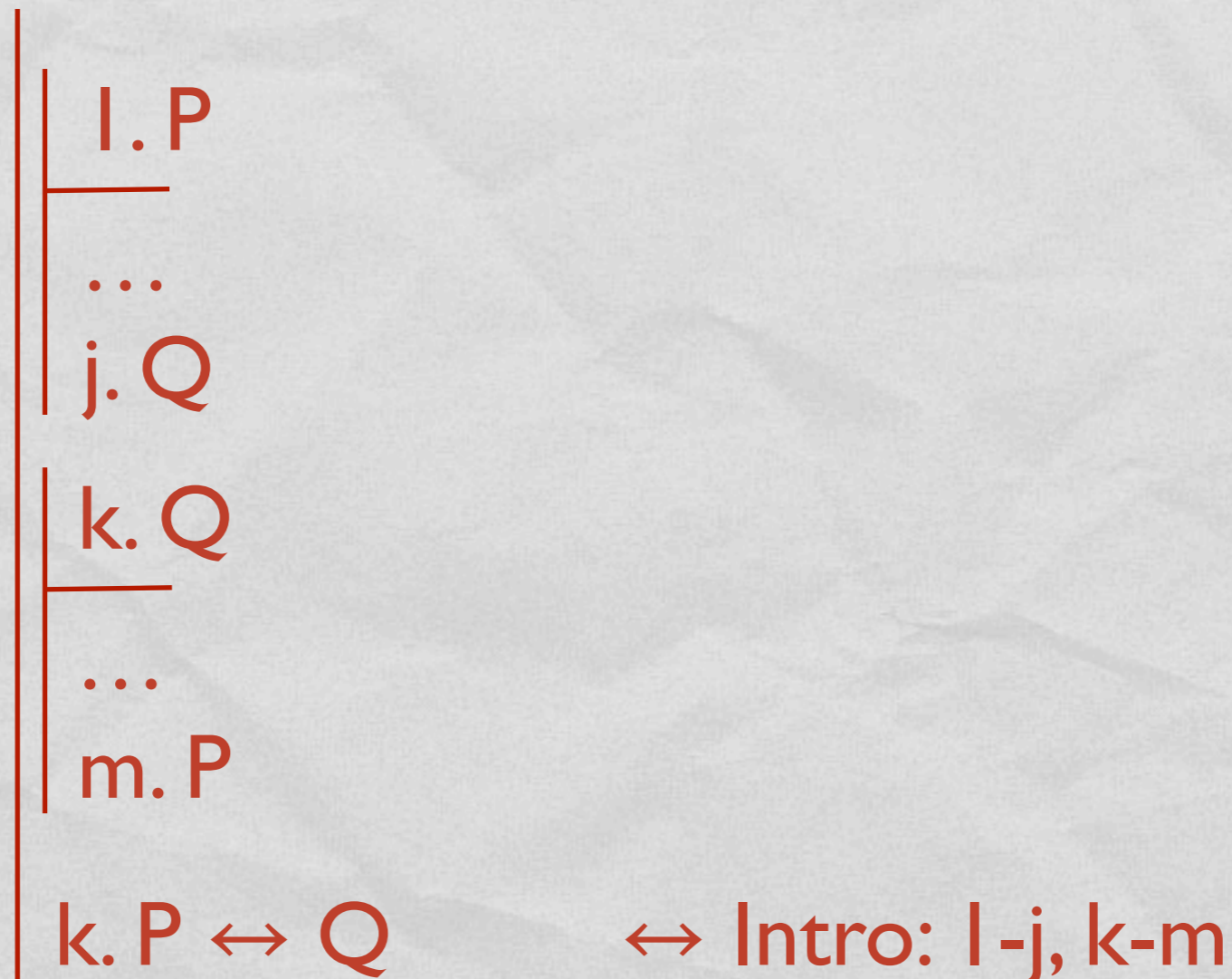
If it is true that if I pray then my prayers will be answered, then there is a God. But I don't pray. Therefore, there is a God.

PROOFS WITH CONDITIONALS 3

Friday, 24 September

FORMAL PROOF RULES

- \leftrightarrow Introduction: from a proof from P to Q and a proof from Q to P , we can infer $P \leftrightarrow Q$.



PARADOXES OF MATERIAL IMPLICATION

Last time we showed:

$$\frac{\neg P \vee Q}{P \rightarrow Q}$$

We did this by showing that:

$$\frac{\neg P}{P \rightarrow Q}$$

and

$$\frac{Q}{P \rightarrow Q}$$

1. $\neg P$	
2. P	for \rightarrow Intro
3. \perp	\perp Intro 1,2
4. Q	\perp Elim 3
5. $P \rightarrow Q$	\rightarrow Intro 2-4

1. Q	
2. P	for \rightarrow Intro
3. Q	Reit 1
5. $P \rightarrow Q$	\rightarrow Intro 2-3

THE OTHER DIRECTION

Example:

$$\frac{P \rightarrow Q}{\neg P \vee Q}$$

$$1. P \rightarrow Q$$

$$2. \neg(\neg P \vee Q) \quad \text{for } \neg\text{Intro}$$

$$3. \neg P \quad \text{for } \neg\text{Intro}$$

$$4. \neg P \vee Q \quad \vee\text{Intro } 3$$

$$5. \perp \quad \perp \text{Intro } 2,4$$

$$6. P \quad \neg\text{Intro } 3-5$$

$$7. Q \quad \rightarrow\text{Elim } 1,6$$

$$8. \neg P \vee Q \quad \vee\text{Intro } 7$$

$$\perp \quad \perp \text{Intro}$$

$$\neg P \vee Q \quad \neg\text{Intro } 2-$$

THE OTHER DIRECTION

Example:

$$\frac{P \rightarrow Q}{\neg P \vee Q}$$

$$1. P \rightarrow Q$$

$$2. \neg(\neg P \vee Q) \quad \text{for } \neg\text{Intro}$$

$$3. \neg P \quad \text{for } \neg\text{Intro}$$

$$4. \neg P \vee Q \quad \vee\text{Intro } 3$$

$$5. \perp \quad \perp \text{Intro } 2,4$$

$$6. P \quad \neg\text{Intro } 3-5$$

$$7. Q \quad \rightarrow\text{Elim } 1,6$$

$$8. \neg P \vee Q \quad \vee\text{Intro } 7$$

$$9. \perp \quad \perp \text{Intro } 2,8$$

$$10. \neg P \vee Q \quad \neg\text{Intro } 2-9$$

PROVING BICONDITIONALS

We have now proved:

$$\left| \frac{P \rightarrow Q}{\neg P \vee Q} \right. \quad \text{and} \quad \left| \frac{\neg P \vee Q}{P \rightarrow Q} \right|$$

Therefore we could prove:

$$\left| \frac{}{(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)} \right|$$

PROVING BICONDITIONALS

$$\frac{}{(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)}$$

I. $P \rightarrow Q$ for \leftrightarrow Intro

$\neg P \vee Q$

$\neg P \vee Q$ for \leftrightarrow Intro

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$ \leftrightarrow Intro

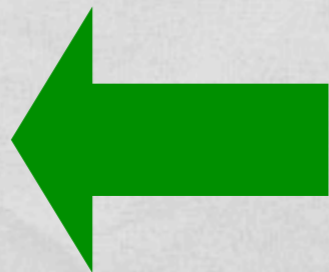
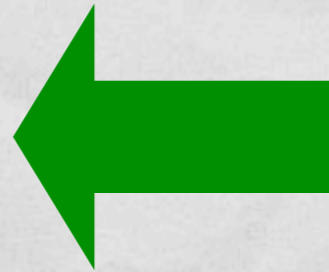
1. $P \rightarrow Q$

$\neg P \vee Q$

$\neg P \vee Q$

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$



1. $P \rightarrow Q$

2. $\neg(\neg P \vee Q)$ for \neg Intro

3. $\neg P$ for \neg Intro

4. $\neg P \vee Q$ \vee Intro 3

5. \perp \perp Intro 2,4

6. P \neg Intro 3-5

7. Q \rightarrow Elim 1,6

8. $\neg P \vee Q$ \vee Intro 7

9. \perp \perp Intro 2,7

10. $\neg P \vee Q$ \neg Intro 2-9

1. $\neg P \vee Q$

2. P for \rightarrow Intro

3. $\neg P$ for \vee Elim

4. \perp \perp Intro 2,3

5. Q \perp Elim 4

6. Q for \vee Elim

7. Q \vee Elim 1,3-5,6-6

8. $P \rightarrow Q$ \rightarrow Intro 2-7

\leftrightarrow Intro 1-10, 11-18

BICONDITIONALS AND EQUIVALENCE

We have now proved:

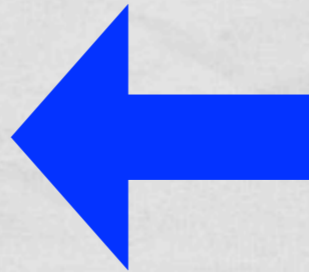
$$\left| \begin{array}{l} \hline (P \rightarrow Q) \leftrightarrow (\neg P \vee Q) \end{array} \right.$$

If a biconditional is a logical truth then the two parts are logically equivalent:

$$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$

CHAINS OF EQUIVALENCE

$$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$



When a conditional is true

By DeMorgan's

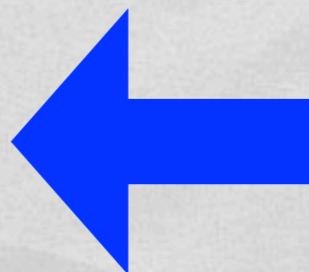
$$(\neg P \vee Q) \Leftrightarrow \neg(P \wedge \neg Q)$$

Therefore

$$(P \rightarrow Q) \Leftrightarrow \neg(P \wedge \neg Q)$$

and so

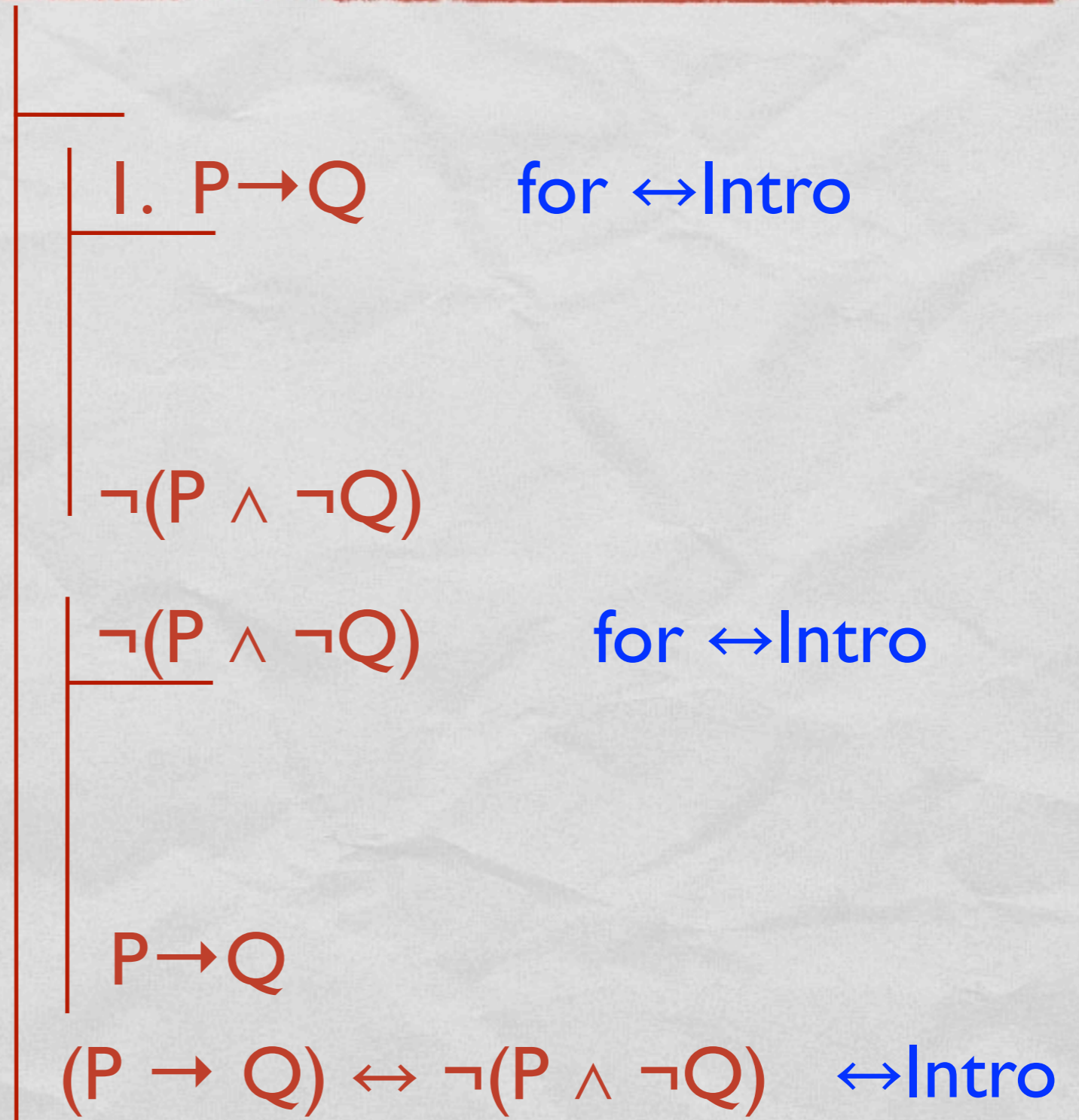
$$\neg(P \rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$$



When a conditional is false

NEGATED CONDITIONALS

$$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$$



1. $P \rightarrow Q$	for \leftrightarrow Intro
2. $P \wedge \neg Q$	for \neg Intro
3. P	\wedge Elim2
4. $\neg Q$	\wedge Elim2
5. Q	\rightarrow Elim 1,3
6. \perp	\perp Intro 4,5
7. $\neg(P \wedge \neg Q)$	\neg Intro 2-6
8. $\neg(P \wedge \neg Q)$	for \leftrightarrow Intro
9. P	for \rightarrow Intro
10. $\neg Q$	for \neg Intro
11. $P \wedge \neg Q$	\wedge Intro 9,10
12. \perp	\perp Intro 8,11
Q	\neg Intro
$P \rightarrow Q$	\rightarrow Intro 9-
$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$	\leftrightarrow Intro

1. $P \rightarrow Q$ for \leftrightarrow Intro
2. $P \wedge \neg Q$ for \neg Intro
3. P \wedge Elim2
4. $\neg Q$ \wedge Elim2
5. Q \rightarrow Elim 1,3
6. \perp \perp Intro 4,5
7. $\neg(P \wedge \neg Q)$ \neg Intro 2-6
8. $\neg(P \wedge \neg Q)$ for \leftrightarrow Intro
9. P for \rightarrow Intro
10. $\neg Q$ for \neg Intro
11. $P \wedge \neg Q$ \wedge Intro 9,10
12. \perp \perp Intro 8,11
13. Q \neg Intro 10-12
14. $P \rightarrow Q$ \rightarrow Intro 9-13
15. $(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$ \leftrightarrow Intro 1-7, 8-14

PUSHING NEGATIONS INSIDE

DeMorgan's Laws

$$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$$

$$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$$

Negated Conditional

$$\neg(P \rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$$

Negated Biconditional

$$\neg(P \leftrightarrow Q) \Leftrightarrow (\neg P \leftrightarrow Q)$$

With repeated applications of these rules, we can convert any sentence with main connective \neg into something with a different main connective.

Or get rid of any particular connectives that we don't like

DOUBLE REDUCTIONS

The Law of the Excluded Middle

$P \vee \neg P$

1. $\neg(P \vee \neg P)$ for \neg -Intro
2. P for \neg -Intro
3. $P \vee \neg P$ \vee -Intro 2
4. \perp \perp -Intro 1,3
5. $\neg P$ \neg -Intro 2-4
6. $P \vee \neg P$ \vee -Intro 5
 \perp
 $P \vee \neg P$ \neg -Intro

DOUBLE REDUCTIONS

The Law of the Excluded Middle

$P \vee \neg P$

1. $\neg(P \vee \neg P)$ for \neg -Intro
2. P for \neg -Intro
3. $P \vee \neg P$ \vee -Intro 2
4. \perp \perp -Intro 1,3
5. $\neg P$ \neg -Intro 2-4
6. $P \vee \neg P$ \vee -Intro 5
7. \perp \perp -Intro 1,6
 $P \vee \neg P$ \neg -Intro 1-7

DOUBLE REDUCTIONS

$A \rightarrow B$
 $\neg A \rightarrow B$
—
 B

1. $A \rightarrow B$
2. $\neg A \rightarrow B$
—
3. $\neg B$ for \neg Intro
—
4. A for \neg Intro
—
5. B \rightarrow Elim 1,3
6. \perp \perp Intro 3,5
7. $\neg A$ \neg Intro 4-6
8. B \rightarrow Elim 2,7
9. \perp \perp Intro 3,8
—
 B \neg Intro 3-9

USING LEM

1. $A \rightarrow B$	
2. $\neg A \rightarrow B$	
3. $\neg B$	for \neg Intro
4. A	for \neg Intro
5. B	\rightarrow Elim 1,3
6. \perp	\perp Intro 3,5
7. $\neg A$	\neg Intro 4-6
8. B	\rightarrow Elim 2,7
9. \perp	\perp Intro 3,8
10. B	\neg Intro 3-9

1. $A \rightarrow B$	
2. $\neg A \rightarrow B$	
3. $A \vee \neg A$	LEM
4. A	for \vee Elim
5. B	\rightarrow Elim 1,3
6. $\neg A$	for \vee Elim
7. B	\rightarrow Elim 2,6
8. B	\vee Elim 3,4-5,6-7

HOW TO REALLY DO PROOFS

6.42 in LPL book

$\neg A \vee \neg(\neg B \wedge (\neg A \vee B))$

1. $\neg(\neg A \vee \neg(\neg B \wedge (\neg A \vee B)))$ for $\neg I$

2. $A \wedge (\neg B \wedge (\neg A \vee B))$ DeMorgans

3. A \wedge Elim

4. $\neg B$ \wedge Elim

5. $\neg A \vee B$ \wedge Elim

6. \perp from 3-5

$\neg A \vee \neg(\neg B \wedge (\neg A \vee B))$

REALLY HARD PROOFS

$$\begin{array}{|l} P \leftrightarrow (Q \leftrightarrow R) \\ \hline (P \leftrightarrow Q) \leftrightarrow R \end{array}$$

This is by no means trivial!
(like it is with \wedge and \vee)

$P \leftrightarrow (Q \leftrightarrow R)$ does NOT mean

$$P \leftrightarrow Q \leftrightarrow R$$

For example, $P \not\leftrightarrow P \leftrightarrow P$

$P \leftrightarrow (P \leftrightarrow P)$ is NOT a tautology