WHICH ARE PROVABLE?

$$\begin{array}{|l} P \rightarrow R \\ \hline (P \wedge Q) \rightarrow R \end{array}$$

$$\begin{array}{|l} P \rightarrow R \\ \hline (P \vee Q) \rightarrow R \end{array}$$

$$\begin{array}{|l} (P \wedge Q) \rightarrow R \\ \hline P \rightarrow R \end{array}$$

$$\begin{array}{|l} (P \vee Q) \rightarrow R \\ \hline P \rightarrow R \end{array}$$

WHICH ARE PROVABLE?

$P \to R$

$(P \wedge Q) \to R$ VALID

$P \to R$

$(P \vee Q) \to R$ INVALID

$(P \wedge Q) \to R$

$P \to R$ INVALID

$(P \vee Q) \to R$

$P \to R$ VALID

PROOFS WITH CONDITIONALS

Wednesday, 22 September

RULES FOR CONDITIONALS

- → Elimination: from P → Q and P, we can infer Q.

  1. P → Q
  2. P
  3. Q → Elim: 1,2

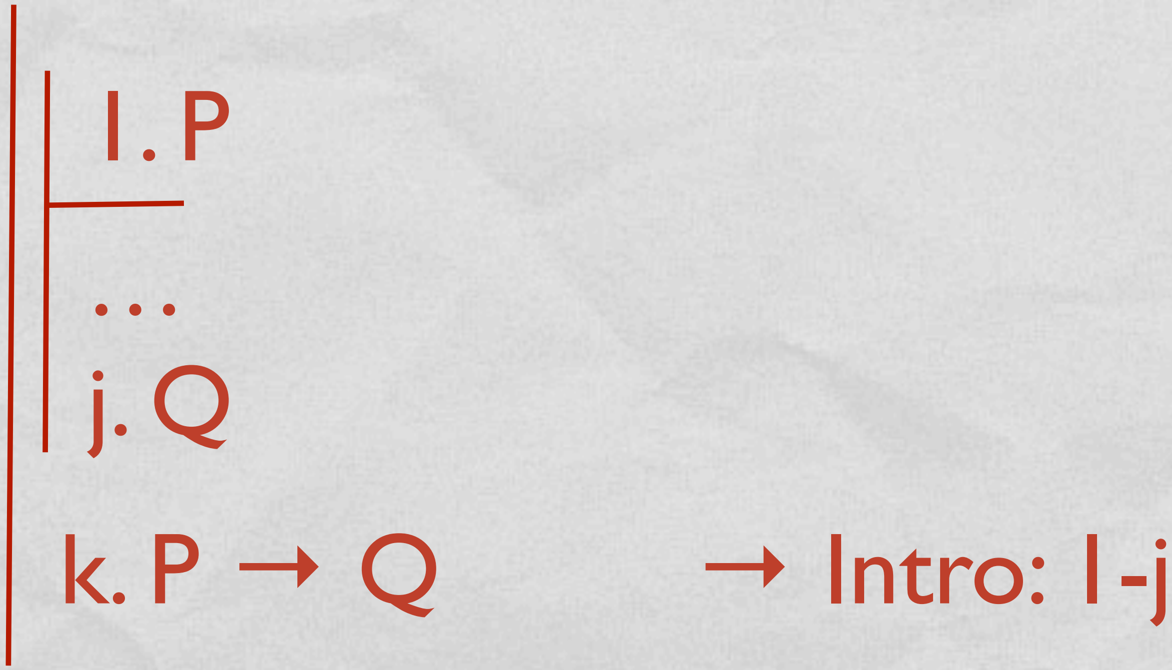
- ↔ Elimination: from P ↔ Q and P/Q, we can infer Q/P.

  1. P ↔ Q
  2. Q
  3. P ↔ Elim: 1,2

FORMAL PROOF RULES

- → Introduction

  From a proof from P to Q, we can infer P → Q.



1. P

…

j. Q

k. P → Q → Intro: 1-j

This rule is often known as Conditional Proof

FORMAL PROOF RULES

- ↔ <u>Introduction</u>: from a proof from P to Q and a proof from Q to P, we can infer P ↔ Q.

1. P

. . .

j. Q

k. Q

. . .

m. P

k. P ↔ Q ↔ Intro: 1-j, k-m

THINK MAIN CONNECTIVE

Example:

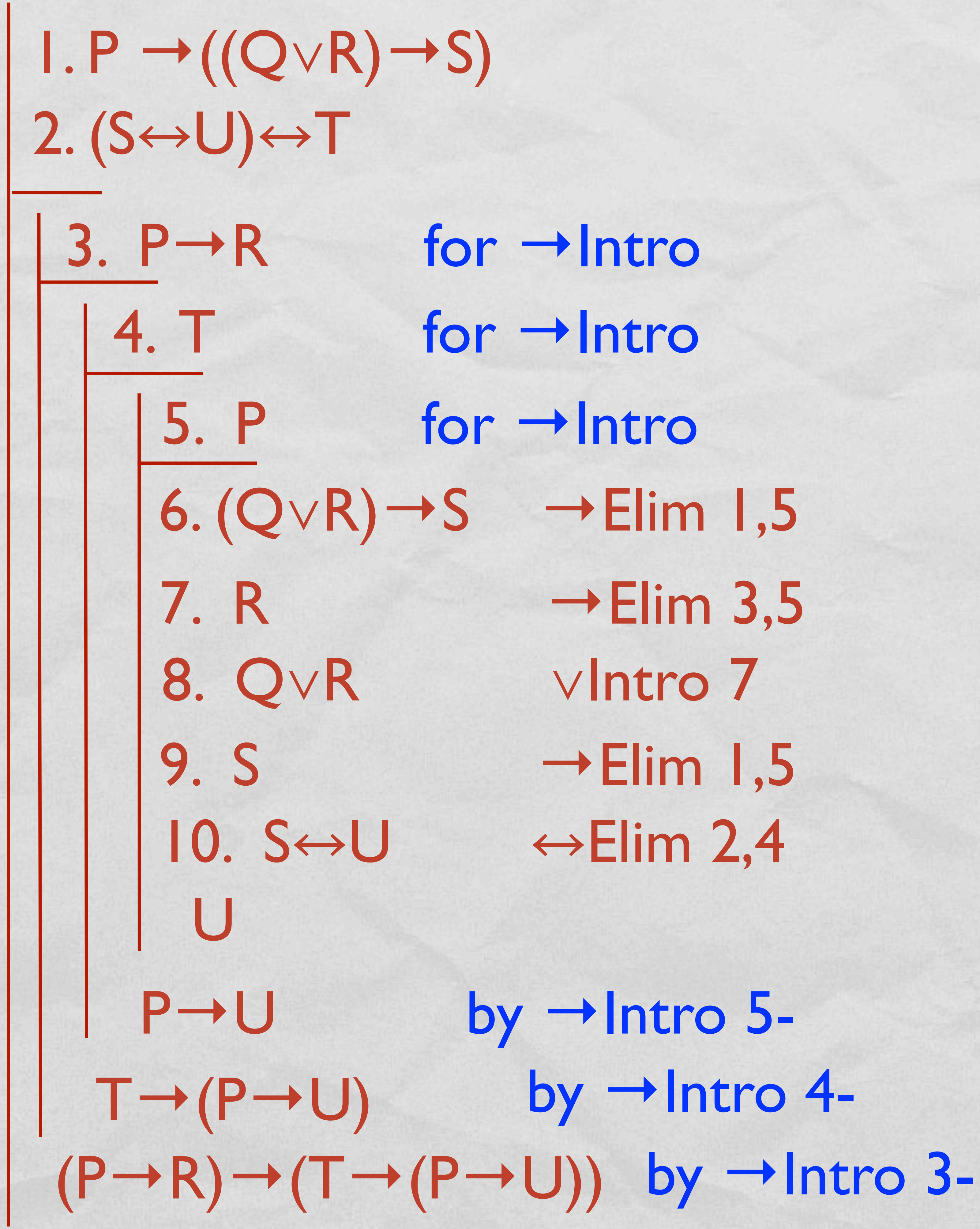
$P \rightarrow ((Q \vee R) \rightarrow S)$

$(S \leftrightarrow U) \leftrightarrow T$

$(P \rightarrow R) \rightarrow (T \rightarrow (P \rightarrow U))$

1. $P \rightarrow ((Q \vee R) \rightarrow S)$
2. $(S \leftrightarrow U) \leftrightarrow T$

$(P \rightarrow R) \rightarrow (T \rightarrow (P \rightarrow U))$



1. $P \rightarrow ((Q \vee R) \rightarrow S)$

2. $(S \leftrightarrow U) \leftrightarrow T$

3. $P \rightarrow R$ for →Intro

4. T for →Intro

5. P for →Intro

6. $(Q \vee R) \rightarrow S$ →Elim 1,5

7. R →Elim 3,5

8. $Q \vee R$ ∨Intro 7

9. S →Elim 1,5

10. $S \leftrightarrow U$ ↔Elim 2,4

U

$P \rightarrow U$ by →Intro 5-

$T \rightarrow (P \rightarrow U)$ by →Intro 4-

$(P \rightarrow R) \rightarrow (T \rightarrow (P \rightarrow U))$ by →Intro 3-

1. $P \to ((Q \lor R) \to S)$
2. $(S \leftrightarrow U) \leftrightarrow T$
3. $P \to R$ for →Intro
4. T for →Intro
5. P for →Intro
6. $(Q \lor R) \to S$ →Elim 1,5
7. R →Elim 3,5
8. $Q \lor R$ ∨Intro 7
9. S →Elim 1,5
10. $S \leftrightarrow U$ ↔Elim 2,4
11. U ↔Elim 9,10
12. $P \to U$ by →Intro 5-11
13. $T \to (P \to U)$ by →Intro 4-12
14. $(P \to R) \to (T \to (P \to U))$ by →Intro 3-13

THINK BACKWARDS

Example:

$(P \rightarrow Q) \rightarrow R$

$S \leftrightarrow \neg Q$

$(P \rightarrow \neg S) \rightarrow R$

1. $(P \rightarrow Q) \rightarrow R$
2. $S \leftrightarrow \neg Q$

3. $P \rightarrow \neg S$ for → Intro

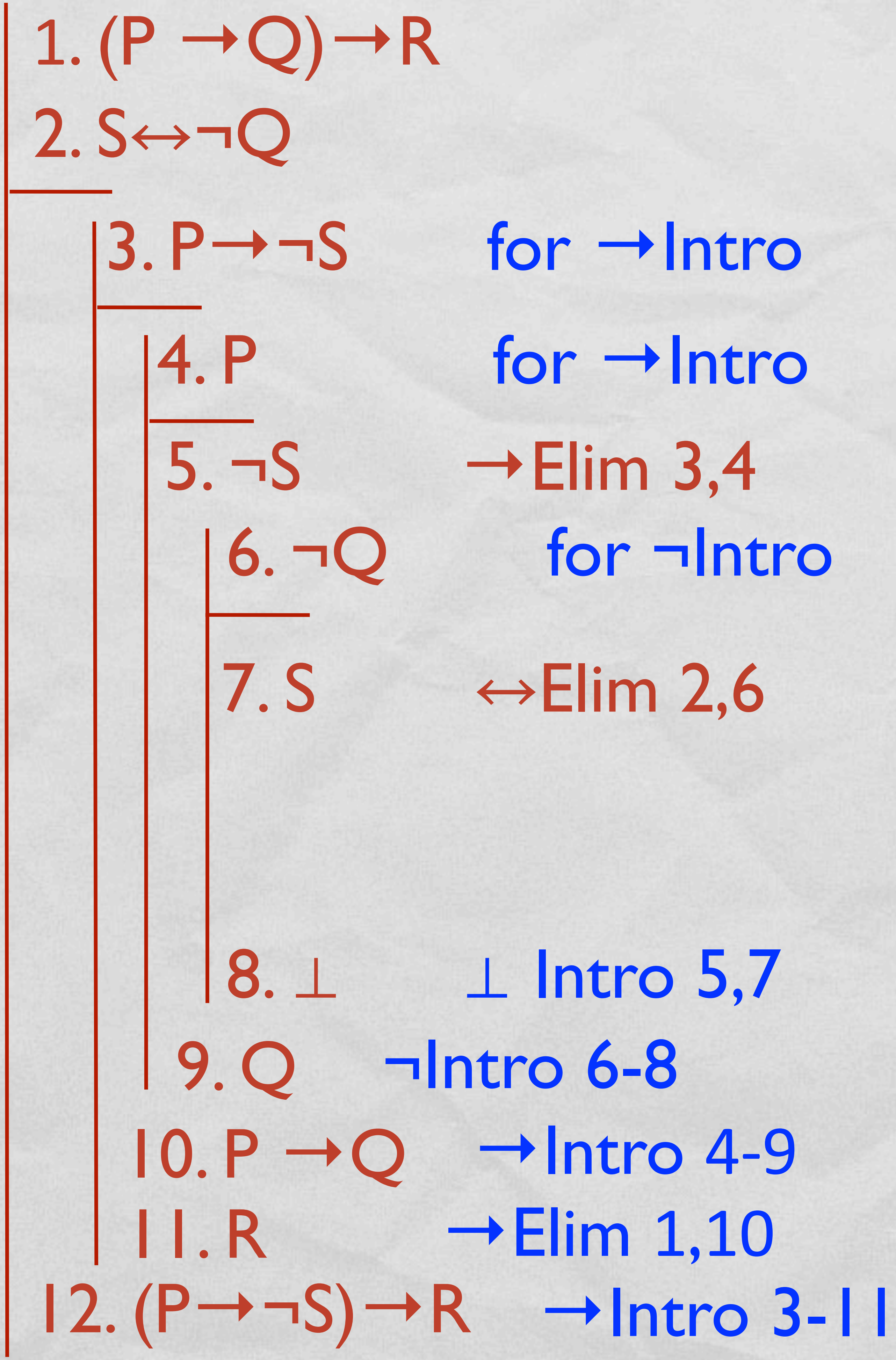
How to get R?
From line 1

$P \rightarrow Q$

R → Elim 1,

$(P \rightarrow \neg S) \rightarrow R$ → Intro 3-

> 1. (P →Q)→R
>
> 2. S↔¬Q
>
> > 3. P→¬S for →Intro
> >
> > > 4. P for →Intro
> > >
> > > 5. ¬S →Elim 3,4
> > >
> > > > 6. ¬Q for ¬Intro
> > > >
> > > > 7. S ↔Elim 2,6
> > > >
> > > > ⊥ ⊥ Intro
> > >
> > > Q ¬Intro 6-
> >
> > P →Q →Intro 4-
> >
> > R →Elim 1,
>
> (P→¬S)→R →Intro 3-



1. (P →Q)→R	
2. S↔¬Q	
3. P→¬S	for →Intro
4. P	for →Intro
5. ¬S	→Elim 3,4
6. ¬Q	for ¬Intro
7. S	↔Elim 2,6
8. ⊥	⊥ Intro 5,7
9. Q	¬Intro 6-8
10. P →Q	→Intro 4-9
11. R	→Elim 1,10
12. (P→¬S)→R	→Intro 3-11

PARADOXES OF MATERIAL IMPLICATION

Example:

¬P ∨ Q

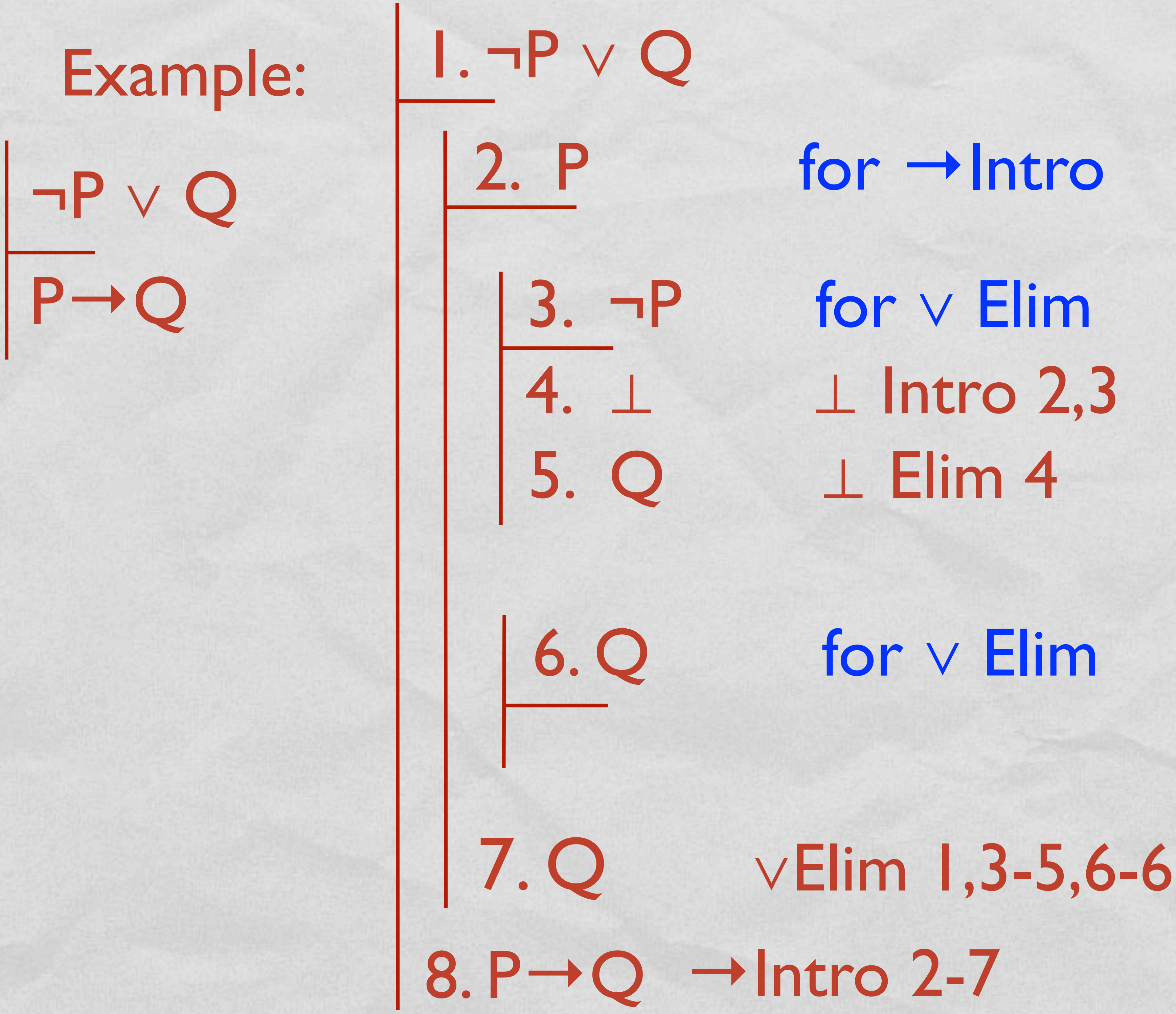
P→Q

1. ¬P ∨ Q	
2. P	for →Intro
3. ¬P	for ∨ Elim
4. ⊥	⊥ Intro 2,3
5. Q	⊥ Elim 4

Now disjunctive syllogism

6. Q	for ∨ Elim
Q	∨Elim 1,3-5,6-6
P→Q	→Intro

PARADOXES OF MATERIAL IMPLICATION



Example:

¬P ∨ Q

P→Q

1. ¬P ∨ Q
    2. P — for →Intro
        3. ¬P — for ∨ Elim
        4. ⊥ — ⊥ Intro 2,3
        5. Q — ⊥ Elim 4
        6. Q — for ∨ Elim
    7. Q — ∨Elim 1,3-5,6-6
8. P→Q — →Intro 2-7

PARADOXES OF MATERIAL IMPLICATION

We just showed:

¬P ∨ Q

P→Q

We did this by showing that:

¬P

P→Q

and

Q

P→Q

1. ¬P	
2. P	for →Intro
3. ⊥	⊥ Intro 1,2
4. Q	⊥ Elim 3
5. P→Q	→Intro 2-4

1. Q	
2. P	for →Intro
3. Q	Reit 1
5. P→Q	→Intro 2-3

THE OTHER DIRECTION

Example:

P → Q

¬P ∨ Q

1. P → Q

2. ¬(¬P ∨ Q) for ¬Intro

3. ¬P for ¬Intro

4. ¬P ∨ Q ∨Intro 3

5. ⊥ ⊥ Intro 2,4

6. P ¬Intro 3-5

7. Q →Elim 1,6

8. ¬P ∨ Q ∨Intro 7

⊥ ⊥ Intro

¬P ∨ Q ¬Intro 2-

The Other Direction

Example:

P → Q

¬P ∨ Q

1. P → Q

> 2. ¬(¬P ∨ Q)   for ¬Intro
>
> > 3. ¬P   for ¬Intro
> >
> > 4. ¬P ∨ Q   ∨Intro 3
> >
> > 5. ⊥   ⊥ Intro 2,4
>
> 6. P   ¬Intro 3-5
>
> 7. Q   →Elim 1,6
>
> 8. ¬P ∨ Q   ∨Intro 7
>
> 9. ⊥   ⊥ Intro 2,8

10. ¬P ∨ Q   ¬Intro 2-9

PROVING BICONDITIONALS

We have now proved:

| P → Q
|---
| ¬P ∨ Q

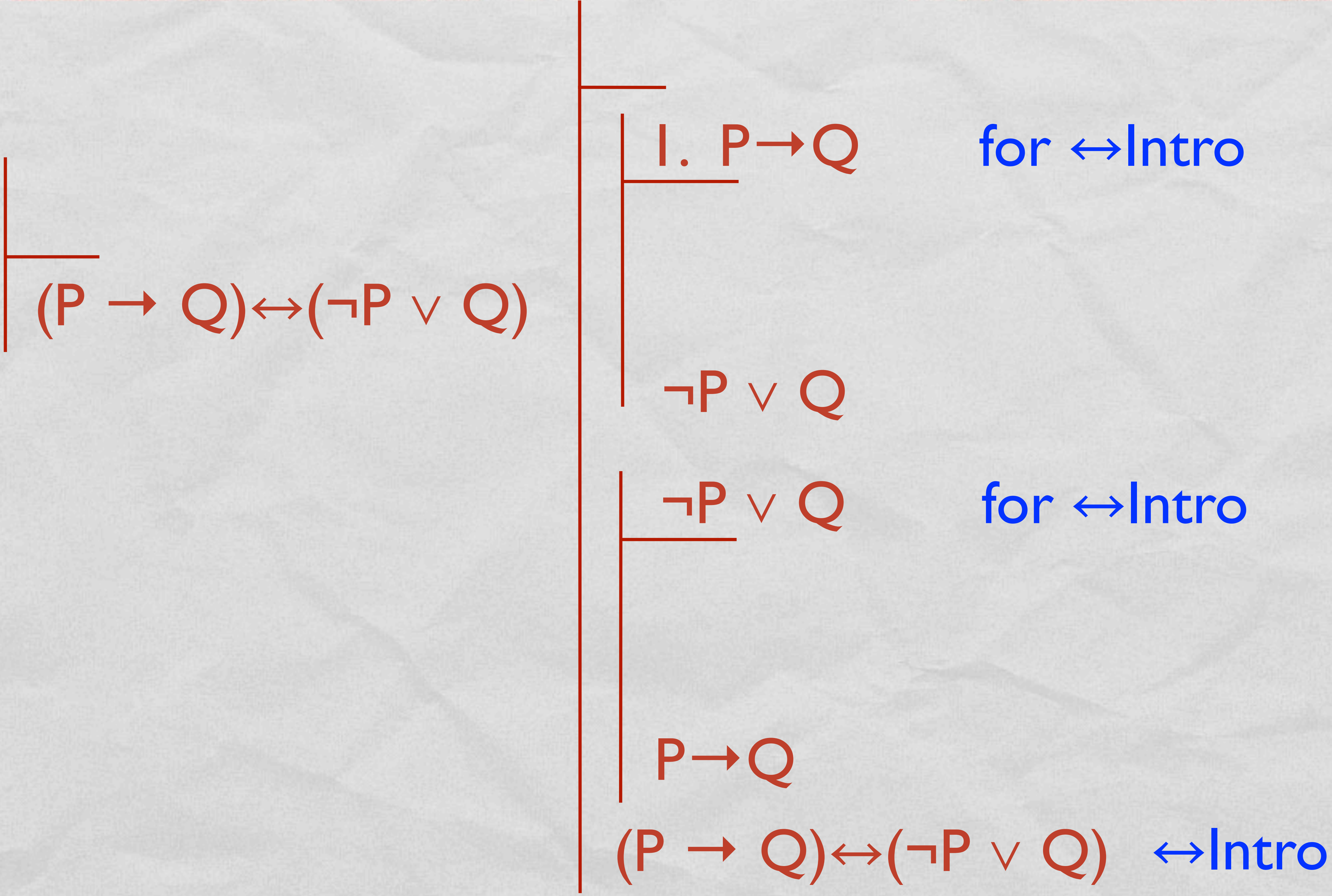
and

| ¬P ∨ Q
|---
| P→Q

Therefore we could prove:

|
|---
| (P → Q)↔(¬P ∨ Q)

PROVING BICONDITIONALS



(P → Q)↔(¬P ∨ Q)

1. P→Q   for ↔Intro

¬P ∨ Q

¬P ∨ Q   for ↔Intro

P→Q

(P → Q)↔(¬P ∨ Q)   ↔Intro

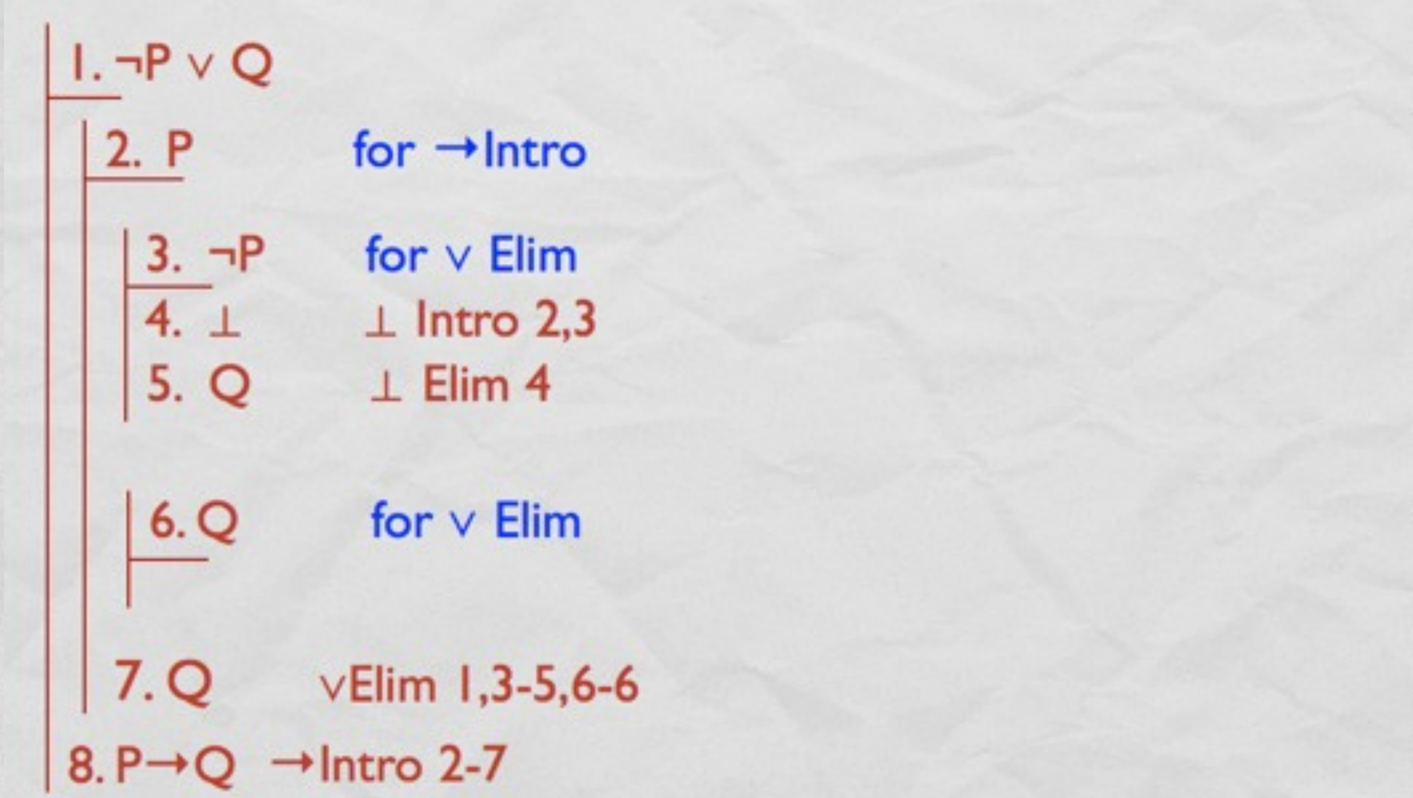
1. P→Q

¬P ∨ Q

¬P ∨ Q

P→Q

1. P → Q
2. ¬(¬P ∨ Q) for ¬Intro
3. ¬P for ¬Intro
4. ¬P ∨ Q ∨Intro 3
5. ⊥ ⊥ Intro 2,4
6. P ¬Intro 3-5
7. Q →Elim 1,6
8. ¬P ∨ Q ∨Intro 7
9. ⊥ ⊥ Intro 2,7
10. ¬P ∨ Q ¬Intro 2-9



1. ¬P ∨ Q
2. P for →Intro
3. ¬P for ∨ Elim
4. ⊥ ⊥ Intro 2,3
5. Q ⊥ Elim 4
6. Q for ∨ Elim
7. Q ∨Elim 1,3-5,6-6
8. P→Q →Intro 2-7

(P → Q)↔(¬P ∨ Q) ↔Intro 1-10, 11-18