You are given four cards with a number on one side and a letter on the other. You can only see one side of each card.

A   B   3   4

Which card(s) do you have to turn over in order to fully test the following rule:

> If there is a vowel on one side of the card,
> then there is an even number on the other side.

You are given four cards with a drink on one side and an age on the other. You can only see one side of each card.

beer	pepsi	16	30

Which card(s) do you have to turn over in order to fully test the following rule:

If you are drinking alcohol, then you must be over 21.

PROOFS WITH CONDITIONALS

Monday, 20 September

Rules for Conditionals

- → Elimination: from $P \rightarrow Q$ and P, we can infer Q.

1.	$P \rightarrow Q$	
2.	P	
3.	Q	→ Elim: 1,2

- ↔ Elimination: from $P \leftrightarrow Q$ and P/Q, we can infer Q/P.

1.	$P \leftrightarrow Q$	
2.	Q	
3.	P	↔ Elim: 1,2

Rules for Conditionals

1. $P \to Q$
2. P

---

3. Q — $\to$ Elim: 1,2

1. $P \to Q$
2. $\neg P$

---

3. $\neg Q$ — INVALID

1. $P \to Q$
2. Q

---

3. P — INVALID

1. $P \to Q$
2. $\neg Q$

---

3. $\neg P$ — VALID, but not $\to$E

Rules for Conditionals

1. $P \rightarrow Q$
2. P

3. Q $\rightarrow$ Elim: 1,2

Modus Ponens

1. $P \rightarrow Q$
2. $\neg Q$

3. $\neg P$ VALID, but not $\rightarrow$E

Modus Tollens

Example

Example:

P ∨ Q

P→R

Q↔¬S

S ∨ R

R

1. P ∨ Q
2. P→R
3. Q↔¬S
4. S ∨ R

---

5. P — for ∨Elim

R

Q — for ∨Elim

R

R — ∨Elim 1, 5-...

Example

Example:

$P \vee Q$

$P \to R$

$Q \leftrightarrow \neg S$

$S \vee R$

R

1. $P \vee Q$
2. $P \to R$
3. $Q \leftrightarrow \neg S$
4. $S \vee R$

---

5. P for $\vee$Elim
6. R $\to$Elim 2,5
7. Q for $\vee$Elim
8. $\neg S$ $\leftrightarrow$Elim 3,7

R by disjunctive syllogism 4,8

R $\vee$Elim 1, 5-...

Rules Using Contradictions

Example: Disjunctive Syllogism

| $P \lor Q$
| $\neg P$
|---
| Q

1. $P \lor Q$	
2. $\neg P$	
3. P	for $\lor$ Elim
4. $\bot$	$\bot$ Intro 2,3
5. Q	$\bot$ Elim 4
6. Q	for $\lor$ Elim
7. Q	$\lor$ Elim 1,3-5,6-6

Rules Using Contradictions

Example: Disjunctive Syllogism

| $P \vee Q$
| $\neg P$
|---
| Q

| 1. $S \vee R$
| 2. $\neg S$
|---
| | 3. S — for $\vee$ Elim
| |---
| | 4. $\bot$ — $\bot$ Intro 2,3
| | 5. R — $\bot$ Elim 4
|
| | 6. R — for $\vee$ Elim
| |---
|
| 7. R — $\vee$ Elim 1,3-5,6-6

Example

1. P ∨ Q	
2. P→R	
3. Q↔¬S	
4. S ∨ R	
5. P	for ∨Elim
6. R	→Elim 2,5
7. Q	for ∨Elim
8. ¬S	↔Elim 3,7
R	by disjunctive syllogism 4,8
R	∨Elim 1, 5-...

Insert DS proof here

3. S	for ∨ Elim
4. ⊥	⊥ Intro 2,3
5. R	⊥ Elim 4
6. R	for ∨ Elim
7. R	∨ Elim 1,3-5,6-6

Example

1. P ∨ Q
2. P→R
3. Q↔¬S
4. S ∨ R

5. P — for ∨Elim
6. R — →Elim 2,5

7. Q — for ∨Elim
8. ¬S — ↔Elim 3,7

Insert DS proof here

9. S — for ∨ Elim
10. ⊥ — ⊥ Intro 8,9
11. R — ⊥ Elim 10

12. R — for ∨ Elim

13. R — ∨ Elim 4,9-11,12-13

13. R — ∨ Elim 4,9-11,12-13

14. R — ∨ Elim 1, 5-6, 7-13

Formal Proof Rules

- → Introduction

  From a proof from P to Q, we can infer P → Q.

  1. P
  ...
  j. Q
  k. P → Q   → Intro: 1-j

This rule is often known as Conditional Proof

Chain Argument

Example:

$P \rightarrow Q$

$Q \rightarrow R$

$P \rightarrow R$

1. $P \rightarrow Q$
2. $Q \rightarrow R$

---

3. P — for $\rightarrow$Intro

---

4. Q — $\rightarrow$ Elim 1,3
5. R — $\rightarrow$ Elim 2,4

6. $P \rightarrow R$ — $\rightarrow$ Intro 3-5

Transitivity of →

Example:

| P → Q
|---
| (Q→R)→(P→ R)

1. P → Q	
2. Q→R	for →Intro
3. P	for →Intro
4. Q	→ Elim 1,3
R	
P → R	→ Intro
(Q → R) → (P → R)	→ Intro

Transitivity of →

Example:

$P \to Q$

$(Q \to R) \to (P \to R)$

1. $P \to Q$
2. $Q \to R$ — for → Intro
3. P — for → Intro
4. Q — → Elim 1,3
5. R — → Elim 2,4
6. $P \to R$ — → Intro 3-5
7. $(Q \to R) \to (P \to R)$ — → Intro 2-6

NOTICE THE STRUCTURE

1. P → Q	
2. Q→R	
3. P	for →Intro
4. Q	→ Elim 1,3
5. R	→ Elim 2,4
6. P → R	→ Intro 3-5

1. P → Q	
2. Q→R	for →Intro
3. P	for →Intro
4. Q	→ Elim 1,3
5. R	→ Elim 2,4
6. P → R	→ Intro 3-5
7. (Q → R) → (P → R)	→ Intro 2-6

Subproofs and Proofs

$P \to Q$

$Q \to R$

___

$P \to R$

$P \to Q$

___

$(Q \to R) \to (P \to R)$

$P \to Q$

$Q \to R$

P

___

R

___

$(P \to Q) \to [(Q \to R) \to (P \to R)]$

Modus Tollens

Example:

P → Q	
¬Q	
¬P	

1. P → Q	
2. ¬Q	
3. P	for ¬Intro
4. Q	→ Elim 1,3
5. ⊥	⊥Intro 2,4
6. ¬P	¬ Intro 3-5

Contraposition

Example:

$$\begin{array}{l|l} & P \rightarrow Q \\ \hline & \neg Q \rightarrow \neg P \end{array}$$

1.	$P \rightarrow Q$	
2.	$\neg Q$	for $\rightarrow$Intro
3.	P	for $\neg$Intro
4.	Q	$\rightarrow$ Elim 1,3
5.	$\perp$	$\perp$Intro 2,4
6.	$\neg P$	$\neg$ Intro 3-5
7.	$\neg Q \rightarrow \neg P$	$\rightarrow$Intro 2-6

Formal Proof Rules

- ↔ Introduction: from a proof from P to Q and a proof from Q to P, we can infer P ↔ Q.

```
|  | l. P
|  |----
|  | ...
|  | j. Q
|  | k. Q
|  |----
|  | ...
|  | m. P
|
| k. P ↔ Q          ↔ Intro: l-j, k-m
```

BICONDITIONALS

Example:

| $P \leftrightarrow Q$
| $Q \leftrightarrow R$
|---
| $P \leftrightarrow R$

1. $P \leftrightarrow Q$	
2. $Q \leftrightarrow R$	
3. P	for ↔Intro
4. Q	→ Elim 1,3
5. R	→ Elim 2,4
6. R	for ↔Intro
7. Q	→ Elim 1,3
8. P	→ Elim 2,4
$P \leftrightarrow R$	↔Intro 3-5, 6-8