Puzzle

You meet A, B, and C in the land of knights and knaves.

A says “Either B and I are both knights or we are both knaves.”

B says “C and I are the same type.”

C says “Either A is a knave or B is a knave.”

Who is what?

METHODS OF PROOF FOR BOOLEAN CONNECTIVES (AND FINISHING TABLES)

Monday, 13 September

Announcements

- Homework 3 due Friday (chapter 7, 4)
- Homework 4 due Monday, Feb 17th (chapter 5,6)
- First in-class exam: Friday, Feb 21
  - Chapters 1-7 (hmws 1-4)

What a truth table can show us

- A sentence is a <u>tautology</u> iff every row of its truth table assigns TRUE to that sentence.
    - A sentence is a <u>contradiction</u> iff it is always false.
- Two sentences are <u>tautologically equivalent</u> iff they have matching truth tables.

WHAT A TRUTH TABLE CAN SHOW US

- A sentence Q is a tautological consequence of a set of sentences $P_1 \ldots P_n$ iff every row of the truth table where $P_1 \ldots P_n$ are all true, Q is also true [i.e. there are NO rows where $P_1 \ldots P_n$ are all true and Q is false].
    - We also say $\{P_1 \ldots P_n\}$ tautologically implies Q
    - This entails that $P_1 \ldots P_n$ therefore Q is a valid argument

Logical and Tautological Consequence

- Example:

  1) A

  2) $A \rightarrow B$

  3) $\neg B \vee C$

  4) Conclusion: C

Logical and Tautological Consequence

A	B	C	A	A→B	¬B ∨ C	C
T	T	T	T	T	F **T**	T
T	T	F	T	T	F **F**	F
T	F	T	T	F	T **T**	T
T	F	F	T	F	T **T**	F
F	T	T	F	T	F **T**	T
F	T	F	F	T	F **F**	F
F	F	T	F	T	T **T**	T
F	F	F	F	T	T **T**	F

No row is T, T, T, F

Logical and Tautological Consequence

A	B	C	A	A→B	B ∨ C	C
T	T	T	T	T	**T**	T
T	T	F	T	T	**T**	F
T	F	T	T	F	**T**	T
T	F	F	T	F	**F**	F
F	T	T	F	T	**T**	T
F	T	F	F	T	**T**	F
F	F	T	F	T	**T**	T
F	F	F	F	T	**T**	F

Second row is T, T, T, F

So NOT valid

Tautological Consequence

- Is this argument valid or not?

  1) $A \rightarrow \neg B$

  2) $B \vee C$

  3) Conclusion: $A \leftrightarrow C$

Lets use Boole - if there is a TTF row then it is invalid

READING THE TABLE

=1= A	=2= B	=3= C	(1) A →	¬B	(2) B ∨ C	(3) A ↔ C
T	T	T	F	F	T	T
T	T	F	F	F	T	F
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	T	F	T	F
F	T	F	T	F	T	T
F	F	T	T	T	T	F
F	F	F	T	T	F	T

TABLES ARE REALLY POWERFUL

- Knights and Knaves problems reduce to a truth table
  - Find the row where these are all true:
    - Knight(a) ↔ ¬Knave(a)
    - Knight(b) ↔ ¬Knave(b)
      - If A says "Both of us are knaves" then add:
    - Knight(a) ↔ [Knave(a)∧Knave(b)]

TABLES ARE REALLY POWERFUL

- Sudoku problems reduce to a truth table
  - Find a row of the table where these are all true:
    - The first cell is exactly one of 1-9:
      - Exactly one of Cell(1,1), Cell(1,2), ..., Cell(1,9)
    - The second cell is 1-9.... the 81st cell is 1-9
    - The first row has exactly one 1:
      - Exactly one of Cell(1,1), Cell(2,1), ..., Cell(9,1)
    - The second row has.... The upper left box has...

Tables are really powerful

- Determining whether (or in which case) a set of sentences can be simultaneously true is sometimes called 'the satisfiability problem' or 'the Boolean satisfiability problem' or '3-sat' (if 3 variables, etc.)
- This problem is EXTREMELY important in computer science because so many problems are equivalent to solving this problem
- But truth tables are trivial (Microsoft Excel will do them for you) so why is this interesting?

Tables are powerful - BUT REALLY SLOW

- In the sudoku case, as written, each sentence is pretty long and there are lots of sentences, but the real problem is the total number of rows. For the 81x9 = 729 variables there are 2^729 rows in the table ≈10^219. My 2.4 GHZ laptop would take ≈10^202 years at maximum efficiency to finish this table.
- Perhaps the most important problem in computer science - Does P=NP?
  - Very roughly equivalent to: Is there a reasonably fast way solve the satisfiability problem?

PROOFS

Why not just use truth tables?

- Truth tables get really HUGE very quickly.
- Truth tables don't mirror the way in which we make arguments.
- Truth tables only show us tautological consequence, for example they are insensitive to identity. We want to capture a broader notion of logical consequence.

Proofs

- We want formal proofs to mirror the kind of reasoning we use informally.
- We will start by looking at some intuitive steps that we use in making valid informal arguments.
- We will then find ways to formalize these steps in our formal system of proof.
- We already have identity introduction (= intro) and identity elimination (= elim).

FORMAL PROOF RULES FOR ∧

- ∧ Introduction

  From P and Q, we can infer P∧Q.

  | 1. P
  | 2. Q
  |———
  | 3. P ∧ Q   ∧ Intro: 1,2

- ∧ Elimination

  From P∧Q, we can infer P.

  | 1. P ∧ Q
  |———
  | 2. P   ∧ Elim: 1

Formal Proof Rules (∧)

Example:

| $A \wedge (B \wedge C)$
|---
| $(A \wedge B) \wedge C$

1. $A \wedge (B \wedge C)$	
2. A	∧ Elim: 1
3. $B \wedge C$	∧ Elim: 1
4. B	∧ Elim: 3
5. C	∧ Elim: 4
6. $A \wedge B$	∧ Intro: 2,4
7. $(A \wedge B) \wedge C$	∧ Intro: 5,6

Main Connectives

- **Incorrect**

1. $\neg(P \rightarrow R)$
2. Q

---

3. $\neg((P \wedge Q) \rightarrow R)$ $\wedge$ Intro: 1,2

- **Incorrect**

1. $\neg(P \wedge Q)$

---

2. $\neg P$ $\wedge$ Elim: 1

PROOFS

<u>Disjunction Introduction</u>

- Intuitively, if you know that A is true, then you can conclude that either A or B (or both).
- Ex: If Alice will be at the party, then it is true that either Alice or Bill will be there.
- In general, from P we can infer 'P or Q'.

Formal Proof Rules (∨)

- ∨ Introduction

  From P, we can infer P∨Q.

1. P	
2. P ∨ Q	∨ Intro: 1

- Another example:

1. P	
2. P∨((Q↔R)→¬S)	∨ Intro: 1

Proof by Cases

- Intuitively, if you know that A or B is the case, and that C follows from A and C also follows from B, then you know that C is the case.
- Example: I will either go to the bank on Monday or Tuesday. So either way, I will have some money to buy lunch on Wednesday.

Proof by Cases

Disjunction Elimination

- In general, proof by cases (disjunction elimination) is when you start with a disjunction and show for each disjunct that, if you assume its truth, some sentence S follows.
- Note: you don't need to know which disjunct is true.

Proof by Cases

- Disjunction Elimination formalizes proof by cases.
- In order to use proof by cases, we need to be able to make assumptions in our proof.
- To show that certain things follow from a set of assumptions, we use subproofs.
- BUT we can only make assumptions within a subproof.

PROOF BY CASES

- ∨ Elimination

  If R follows from P, and if R follows from Q, then from P∨Q, we can infer R.

Scope Lines

Scope Lines indicate assumptions that don't necessarily follow from earlier assumptions

1. P ∨ Q	
2. P	
…	
j. R	??
k. Q	
…	
m. R	??
n. R	∨Elim: 1,2-j,k-m

Proof by Cases

Example:

$(A \wedge B) \vee (A \wedge C)$

A

1. $(A \wedge B) \vee (A \wedge C)$
2. $A \wedge B$
3. A — $\wedge$ Elim: 2
4. $A \wedge C$
5. A — $\wedge$ Elim: 4
6. A — $\vee$ Elim: 1,2-3,4-5