Questions 1–6

A crayon manufacturer is designing a box with ten colors of crayons—M, N, O, P, R, S, T, V, W, and Y. The crayons will be arranged in two rows—front and back—and five columns, labeled 1–5 from left to right. The following conditions apply:

P is in the same row as O, and there are exactly three crayons between them.

S is in the same column as W.

T is directly to the left of P.

If M is in the front row, W and Y are in the back row.

If W is in the front row, O is in the back row.

R is in the third column.

1. Which one of the following could be a possible arrangement of crayons, listed from left to right?

(A) Front: O, S, R, T, P
Back: W, N, M, Y, V

(B) Front: O, S, R, T, P
Back: N, W, V, Y, M

(C) Front: Y, W, N, R, V
Back: P, S, M, T, O

(D) Front: O, R, S, T, P
Back: N, V, W, Y, M

(E) Front: N, W, Y, V, M
Back: O, S, R, T, P

2. If O is in the first column of the front row, which one of the following must be true?

(A) R is in the third column of the front row.
(B) M is in the third column of the front row.
(C) M is in the third column of the back row.
(D) V is in the fifth column of the back row.
(E) W is in the second column of the back row.

3. If W is in the front row, how many exact positions of crayons can be determined?

(A) 4
(B) 5
(C) 6
(D) 7
(E) 8

4. If M is in the front row, it must be true that

(A) O is in the front row
(B) S is in the front row
(C) T is in the back row
(D) R is in the front row
(E) R is in the back row

If V and Y are in the front row, which one of the following could be true?

(A) S is in the front row.
(B) O is in the front row.
(C) T is in the front row.
(D) M is in the front row.
(E) R is in the back row.

6. Which one of the following conditions, if true, would determine the complete order for at least one of the rows?

(A) T is in the fourth column in the back row.
(B) V is in the fourth column of the front row.
(C) S is in the second column in the front row.
(D) W is in the second column in the front row.
(E) P is in the fifth column of the front row.

Diagrams and Validity

Wednesday, 30 April

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- An argument is FO-valid if any interpretation that makes all of the premises true also makes the conclusion true.

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INTERPRETATIONS

- An argument is FO-valid if any interpretation that makes all of the premises true also makes the conclusion true.
  - So it is invalid if there is at least one interpretation that makes all the premises true and makes the conclusion false.
- An *interpretation* gives the meaning of the constants, functions, and predicates and gives a domain (so we know what 'for all x' means). - it gives enough information to know whether any particular sentence is true or false.

Interpretations

$\exists x\, P(x) \wedge \exists x\, Q(x) \nvdash \exists x(P(x) \wedge Q(x))$ - it is FO invalid

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A picture with one cube and one tet

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A picture with one cube and one tet

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But the other direction is correct

$$\exists x(P(x) \wedge Q(x)) \vdash \exists x\, P(x) \wedge \exists x\, Q(x)$$

valid or not?

1. $\exists x \exists y(Square(x) \land Square(y) \land LeftOf(x,y))$
2. $\forall x \forall y(LeftOf(x,y) \rightarrow \neg Filled(x))$

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3. $\forall x \forall y((Filled(x) \land Filled(y)) \rightarrow SameRow(x,y))$

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2. $\forall x \forall y(\text{LeftOf}(x,y) \rightarrow \neg \text{Filled}(x))$

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3. $\forall x \forall y((\text{Filled}(x) \land \text{Filled}(y)) \rightarrow \text{SameRow}(x,y))$

Try to make the premises true and the conclusion false. To do this, first add things to make the ∃x s true (in the premises) and to make the ∀x s false (if in the conclusion).

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Try to make the premises true and the conclusion false. To do this, first add things to make the $\exists x$ s true (in the premises) and to make the $\forall x$ s false (if in the conclusion).

Ans: Picture on board in class - a non-filled square left of two filled squares which aren't on the same row (for example - other pictures work).

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You could try to give meanings to T, M, and A and then use Tarski's world or a shapes diagram, but it is usually easier (and safer) to draw a diagram.

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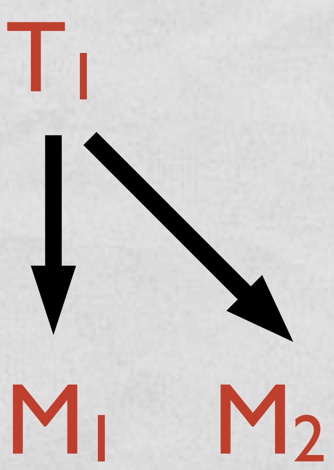
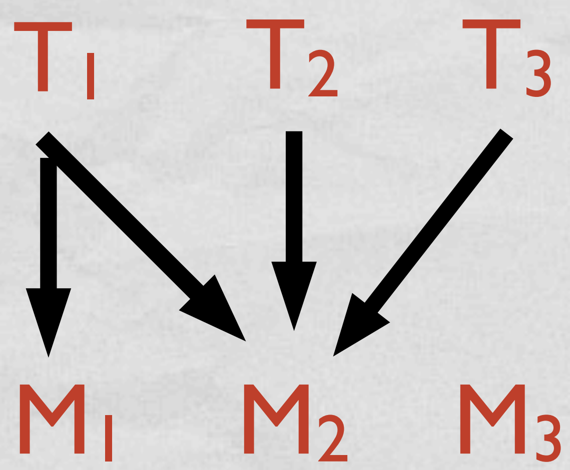
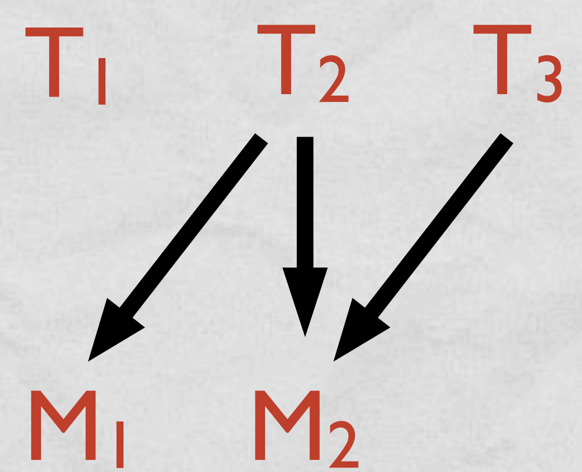
Label the Ts and the Ms and then have A(x,y) = x points to y

Examples







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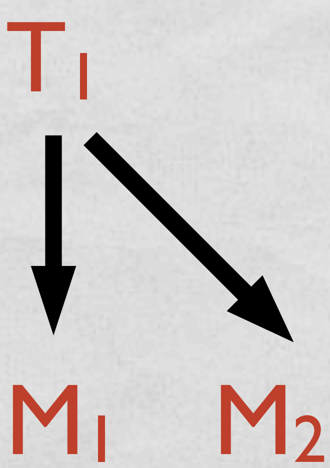
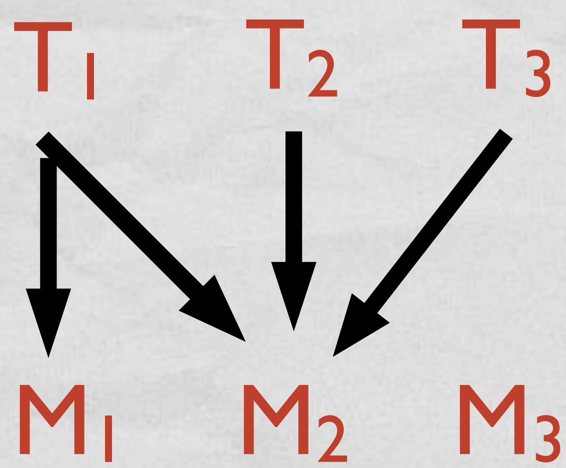
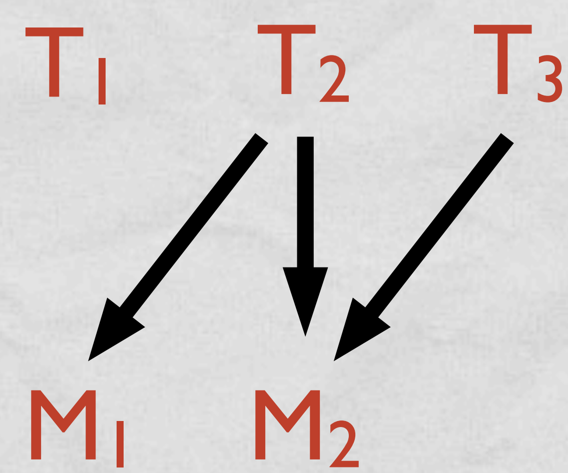
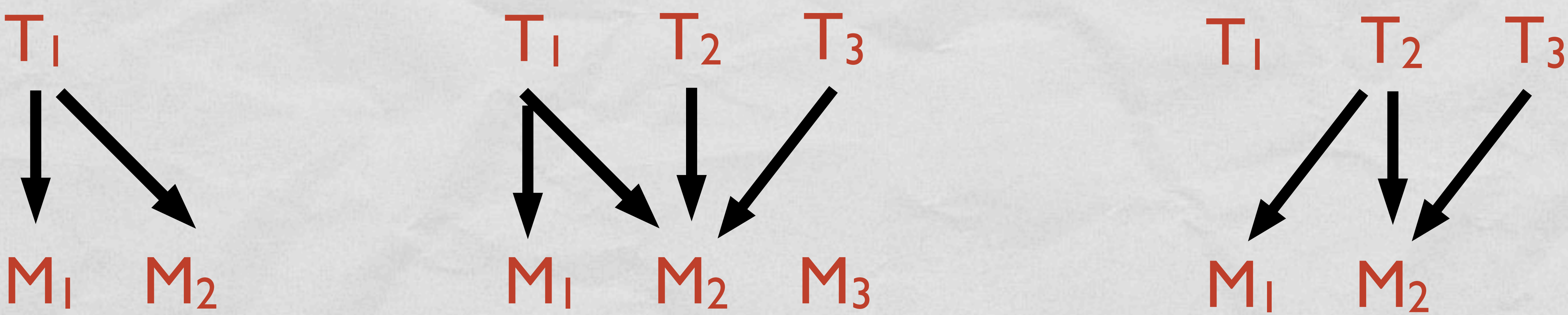






$$\exists x(T(x) \wedge \forall y(M(y) \rightarrow A(x,y)))$$

Examples



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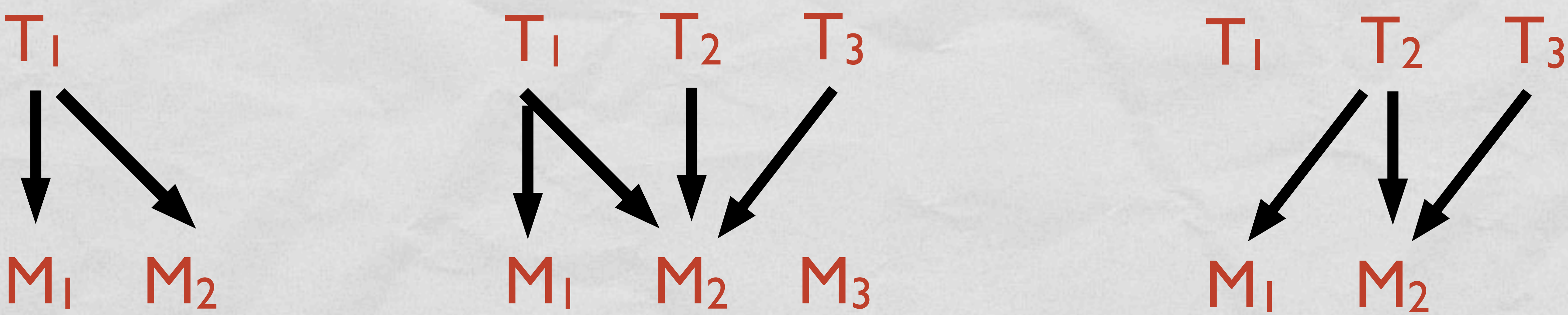
True, False, True

Examples



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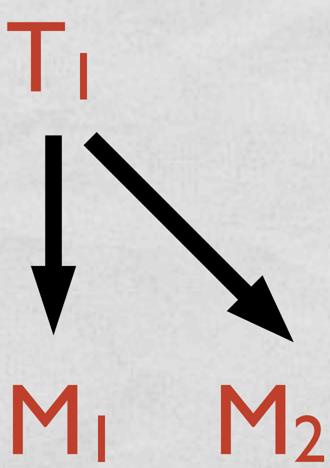

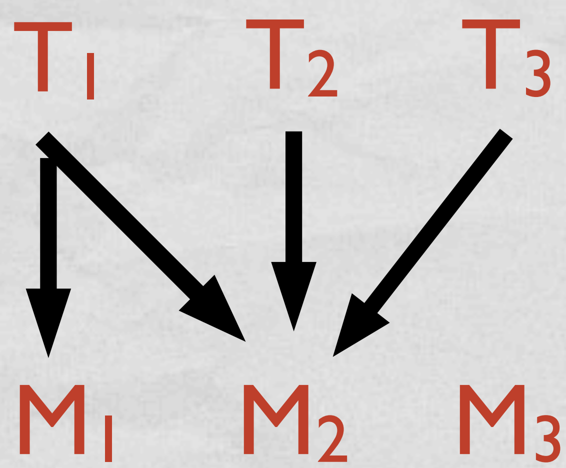

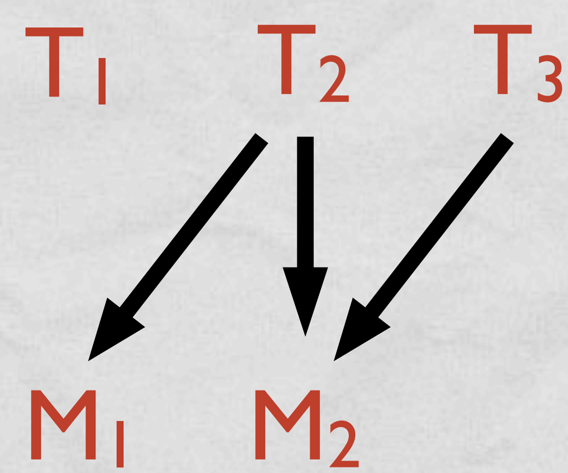

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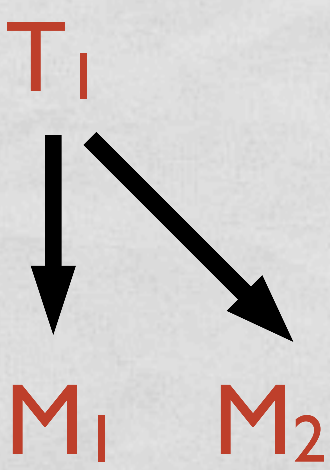

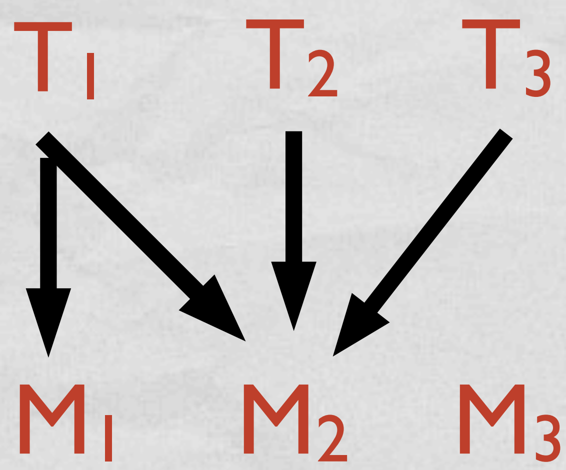

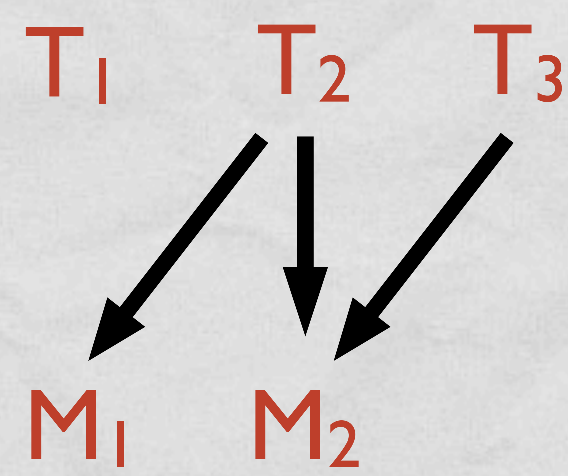

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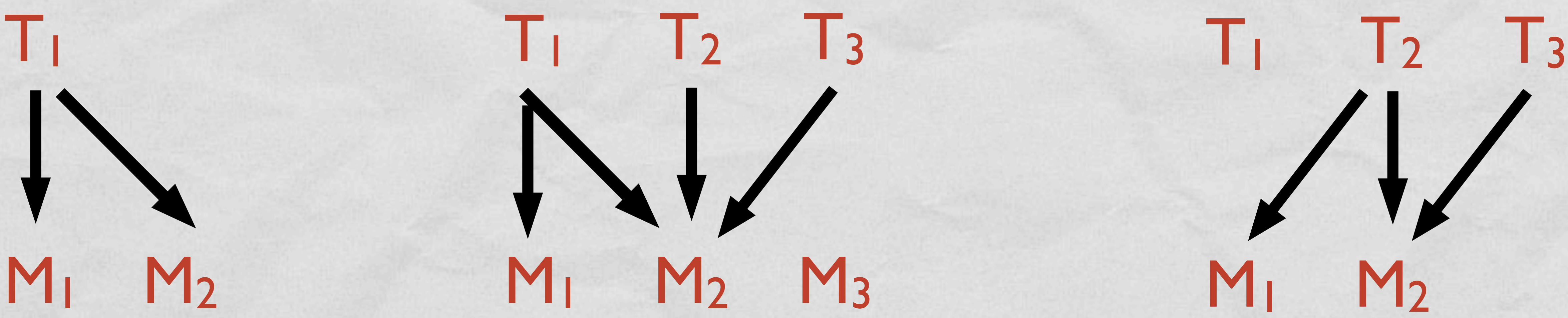

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2. $\exists x(M(x) \wedge \forall y(T(y) \to A(y,x))$

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valid or not?

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Lets try to show this is invalid using a diagram. We are trying to make both of the premises true and the conclusion false.

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For P2, we need an M that every T points to. - Lets call it M_1

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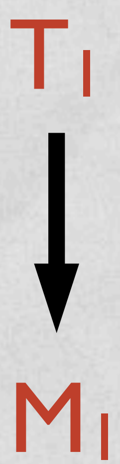

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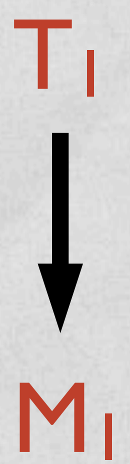

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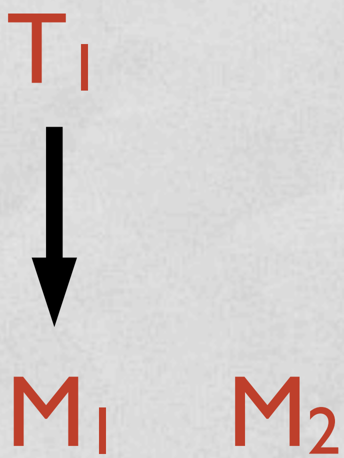

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For P1, we need to make sure that for every T, there is at least one M that it points to. Right now, that is true. So we are done.

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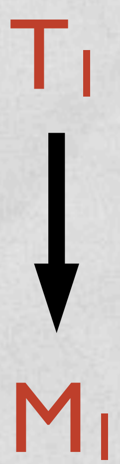

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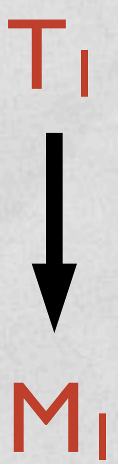

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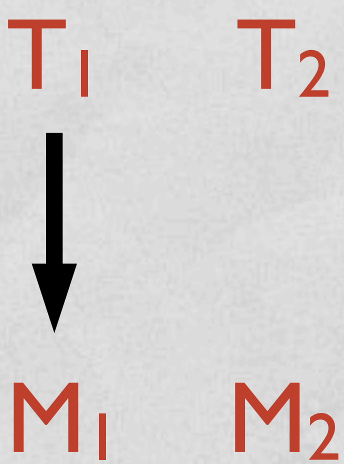

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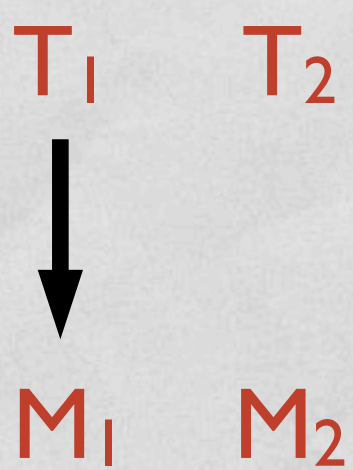

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But don't forget T_1 was supposed to point to every M. Every time you add to the picture, make sure you keep that true.

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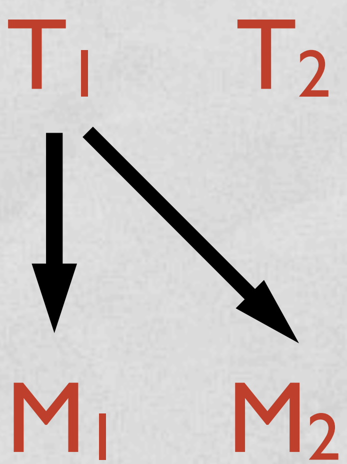

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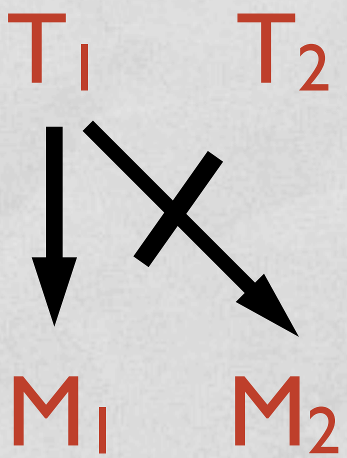

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***But you can't do that here. M_2 was the meeting that nothing went to. Adding an extra meeting won't help either. So it is valid.

Making a Universal False

$\forall x(M(x) \rightarrow \exists y(T(y) \wedge A(y,x))$ -- How to make it false?

NEGATED QUANTIFIERS

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- $\forall x P(x)$ is like a big conjunction.

Negated Quantifiers

- $\forall x P(x)$ is like a big conjunction.
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By DeMorgan's like thinking....

- $\neg\forall x P(x) \Leftrightarrow \exists x \neg P(x)$ (a big disjunction of negations)

NEGATED QUANTIFIERS

- $\forall x P(x)$ is like a big conjunction.
- $\neg \forall x P(x)$ is like the negation of a big conjunction.

By DeMorgan's like thinking....

- $\neg \forall x P(x) \Leftrightarrow \exists x \neg P(x)$ (a big disjunction of negations)

By the same thought....

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By the same thought....

- $\neg\exists x P(x) \Leftrightarrow \forall x \neg P(x)$

Making a Universal False

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To make an existential true, add something to the picture.

Making a Universal False

To make an existential true, add something to the picture.

To make an existential false, don't add anything.

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To make a universal true, you have to make sure that each time you add something to the picture, you go back and check the universal.

Making a Universal False

To make an existential true, add something to the picture.

To make an existential false, don't add anything.

To make a universal true, you have to make sure that each time you add something to the picture, you go back and check the universal.

To make a universal false, create a counterexample.

Making a Universal False

To make an existential true, add something to the picture.

To make an existential false, don't add anything.

To make a universal true, you have to make sure that each time you add something to the picture, you go back and check the universal.

To make a universal false, create a counterexample.

To make $\forall x P(x)$ false, make $\exists x \neg P(x)$ true.

Making a Universal False

$\forall x(Cube(x) \rightarrow Small(x))$ -- How to make it false?

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$\neg\forall x(Cube(x) \rightarrow Small(x)) \Leftrightarrow$

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$\exists x\neg(Cube(x) \rightarrow Small(x)) \Leftrightarrow$ (negate the quantifier)

Making a Universal False

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$\exists x\neg(Cube(x) \rightarrow Small(x)) \Leftrightarrow$ (negate the quantifier)

$\exists x(Cube(x) \wedge \neg Small(x)) \Leftrightarrow$ (by taut con)

Making a Universal False

$\forall x(Cube(x) \rightarrow Small(x))$ -- How to make it false?

$\neg\forall x(Cube(x) \rightarrow Small(x)) \Leftrightarrow$

$\exists x\neg(Cube(x) \rightarrow Small(x)) \Leftrightarrow$ (negate the quantifier)

$\exists x(Cube(x) \wedge \neg Small(x)) \Leftrightarrow$ (by taut con)

So there is a cube which is not small

MAKING A UNIVERSAL FALSE

$\forall x(M(x) \rightarrow \exists y(T(y) \wedge A(y,x))$ -- How to make it false?

Making a Universal False

$\forall x(M(x) \rightarrow \exists y(T(y) \wedge A(y,x))$ -- How to make it false?

$\neg\forall x(M(x) \rightarrow \exists y(T(y) \wedge A(y,x))$ $\Leftrightarrow$

Making a Universal False

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Making a Universal False

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$\neg\forall x(M(x) \rightarrow \exists y(T(y) \wedge A(y,x))$ $\Leftrightarrow$

$\exists x\neg(M(x) \rightarrow \exists y(T(y) \wedge A(y,x))$ $\Leftrightarrow$ (negate the quantifier)

$\exists x(M(x) \wedge \neg\exists y(T(y) \wedge A(y,x))$ $\Leftrightarrow$ (by taut con)

Making a Universal False

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$\exists x\neg(M(x) \rightarrow \exists y(T(y) \wedge A(y,x))$ $\Leftrightarrow$ (negate the quantifier)

$\exists x(M(x) \wedge \neg\exists y(T(y) \wedge A(y,x))$ $\Leftrightarrow$ (by taut con)

So there is an M where it is false that there is a T that points to it. -- So no T points to it.

Making a Universal False

$\forall x(M(x) \rightarrow \exists y(T(y) \wedge A(y,x))$ -- How to make it false?

$\neg\forall x(M(x) \rightarrow \exists y(T(y) \wedge A(y,x))$ $\Leftrightarrow$

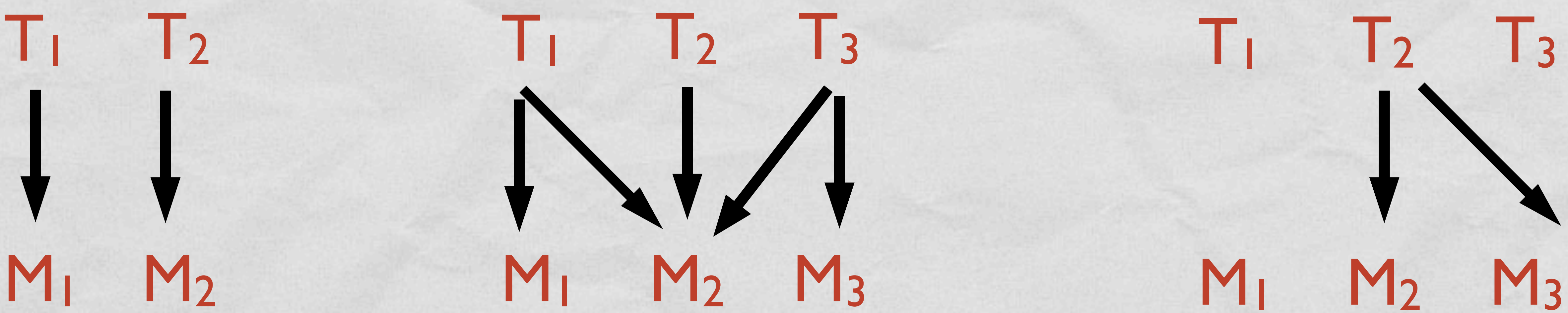
$\exists x\neg(M(x) \rightarrow \exists y(T(y) \wedge A(y,x))$ $\Leftrightarrow$ (negate the quantifier)

$\exists x(M(x) \wedge \neg\exists y(T(y) \wedge A(y,x))$ $\Leftrightarrow$ (by taut con)

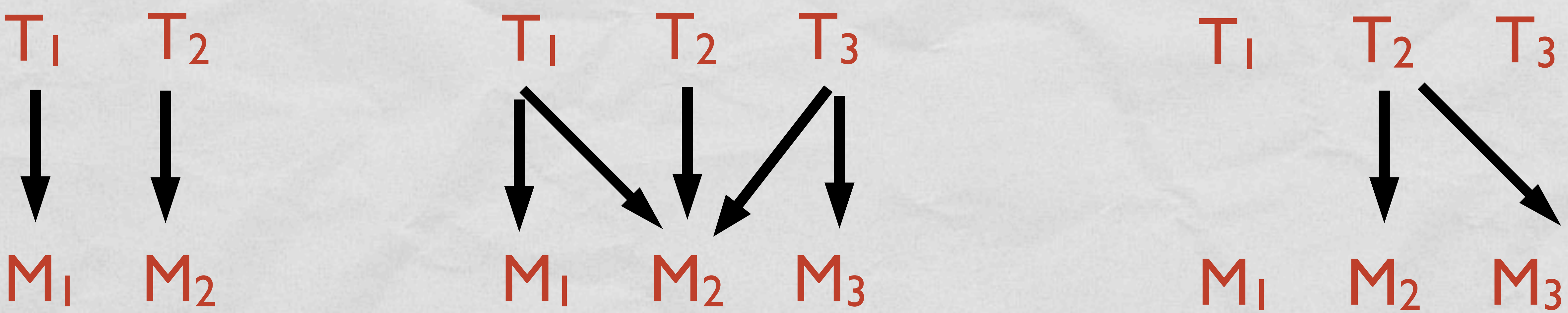
So there is an M where it is false that there is a T that points to it. -- So no T points to it.

$\exists x(M(x) \wedge \forall y(T(y) \rightarrow \neg A(y,x))$ (quantifer + taut con)

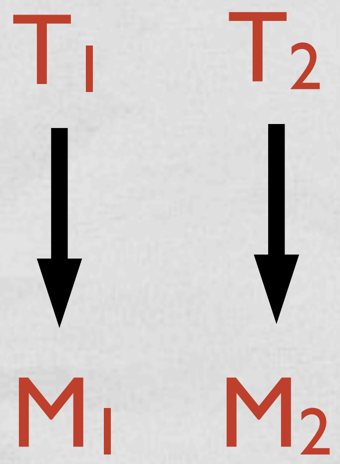
Examples

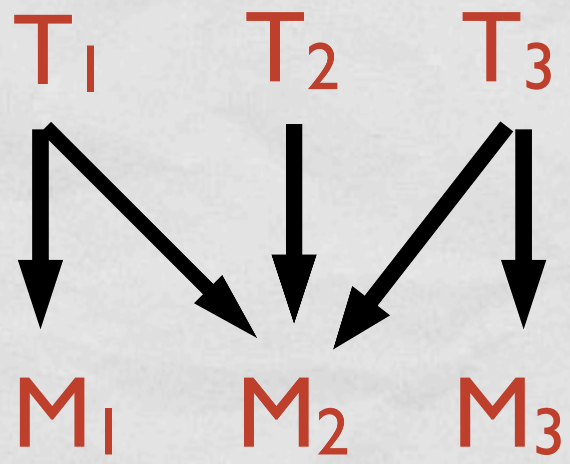


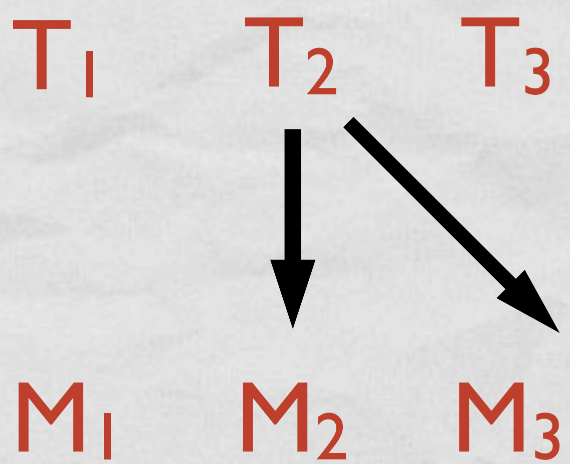
EXAMPLES

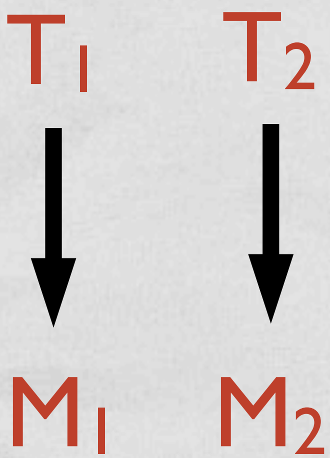


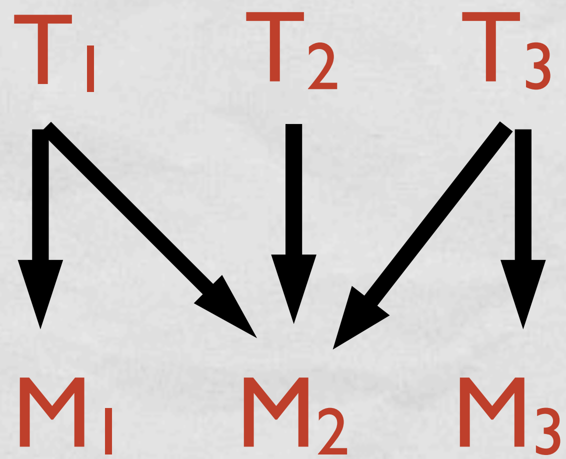
A very natural thing you might want to do is to talk not just about single teachers or single meetings, but about pairs of teachers or pairs of meetings. E.g. there is a pair of teachers who went to exactly the same meetings. Or a pair of meetings that between the two, every teacher went to.

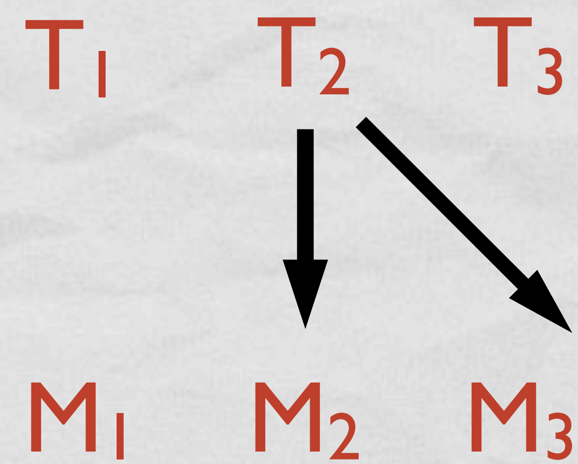

T1
T2
M1
M2


T1
T2
T3
M1
M2
M3

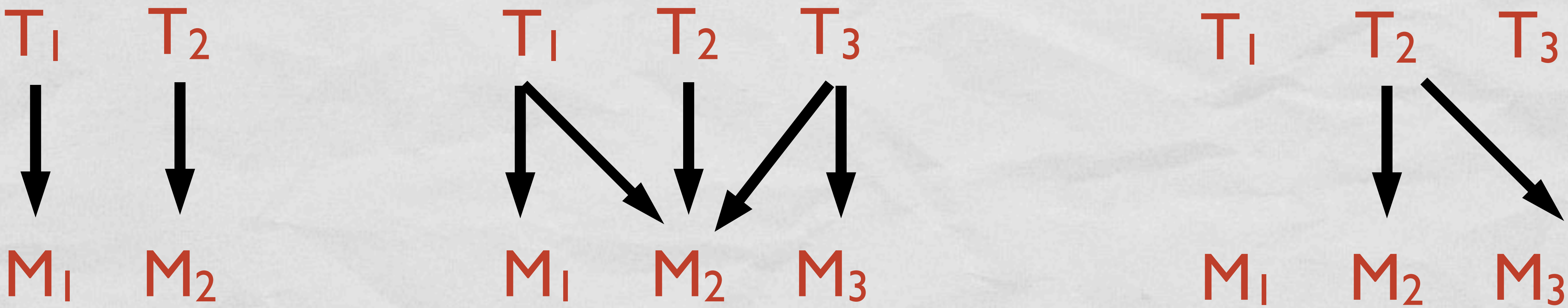

T1
T2
T3
M1
M2
M3





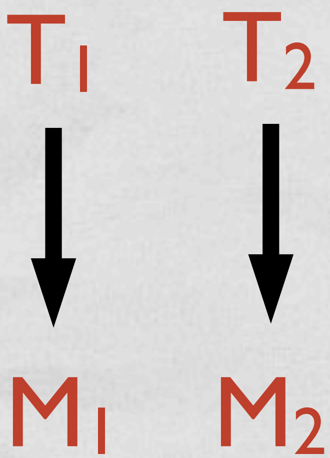


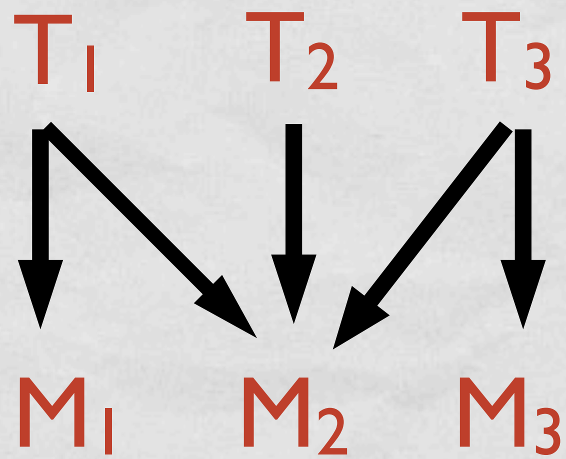
$\exists x \exists y (T(x) \wedge T(y) \wedge \forall z (M(z) \rightarrow (A(x,z) \vee A(y,z))))$

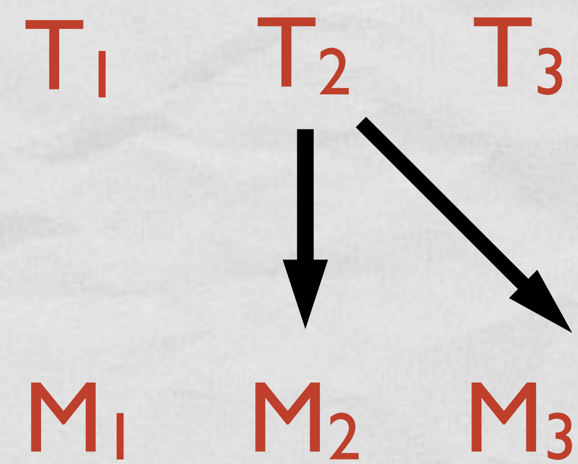


$\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \vee A(y,z))))$

There is a pair of Ts such that for every M, at least one of those Ts went.



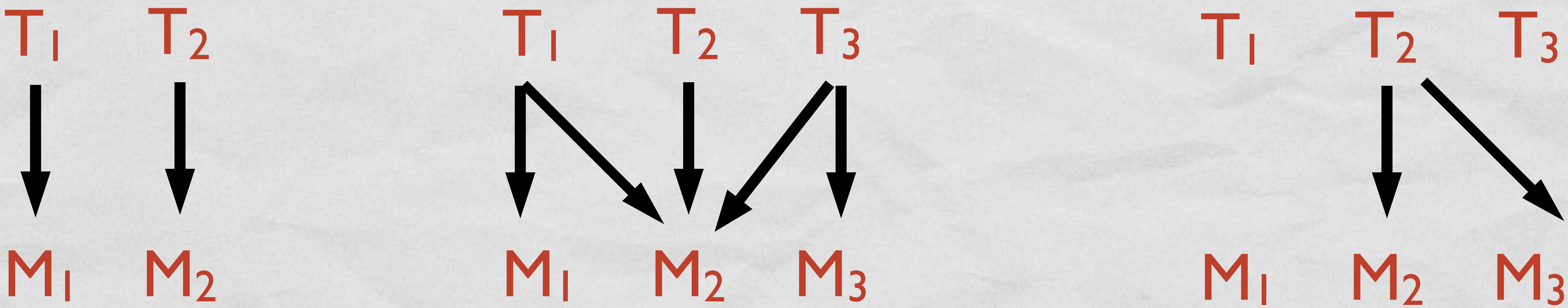




$$\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \vee A(y,z))))$$

There is a pair of Ts such that for every M, at least one of those Ts went.

T, T, F

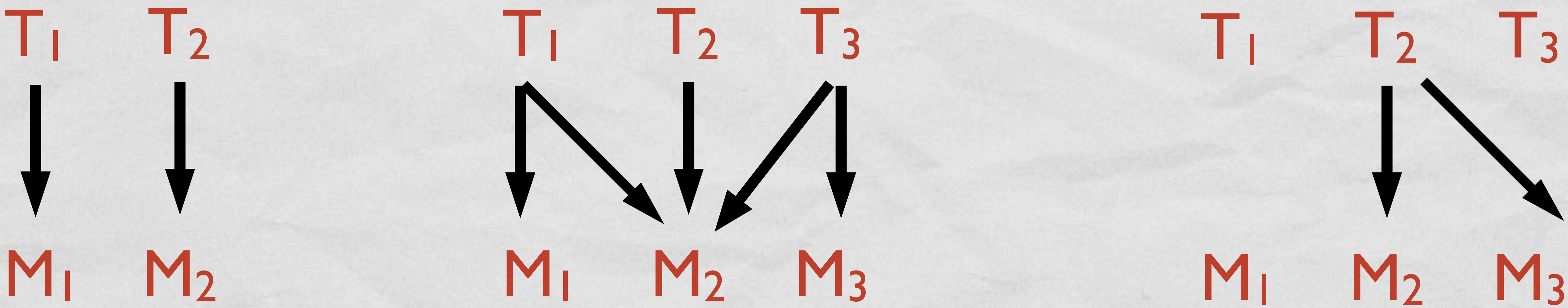


$\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \vee A(y,z))))$

There is a pair of Ts such that for every M, at least one of those Ts went.

T, T, F

$\forall x \forall y((M(x) \wedge M(y)) \rightarrow \exists z(T(z) \wedge (A(z,x) \leftrightarrow A(z,y))))$



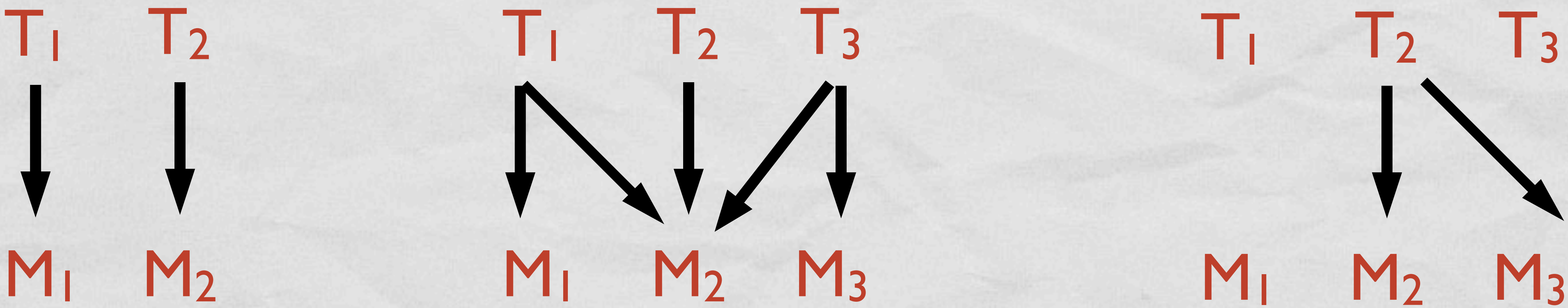
$\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \vee A(y,z))))$

There is a pair of Ts such that for every M, at least one of those Ts went.

T, T, F

$\forall x \forall y((M(x) \wedge M(y)) \rightarrow \exists z(T(z) \wedge (A(z,x) \leftrightarrow A(z,y))))$

For every pair of Ms, there is a T that went to the first if and only if they went to the second.

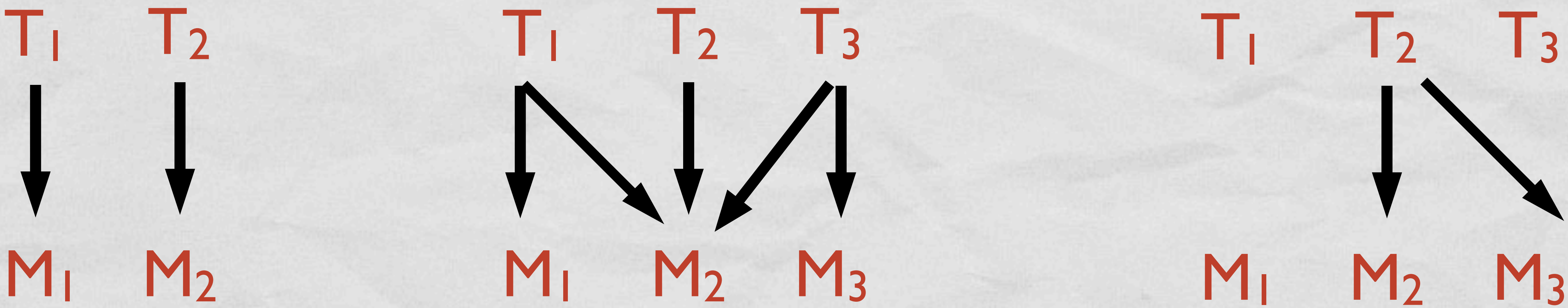


$\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \vee A(y,z))))$

There is a pair of Ts such that for every M, at least one of those Ts went. T, T, F

$\forall x \forall y((M(x) \wedge M(y)) \rightarrow \exists z(T(z) \wedge (A(z,x) \leftrightarrow A(z,y))))$

For every pair of Ms, there is a T that went to the first if and only if they went to the second. F, T, T



$\exists x \exists y(T(x) \wedge T(y) \wedge \forall z(M(z) \rightarrow (A(x,z) \vee A(y,z))))$

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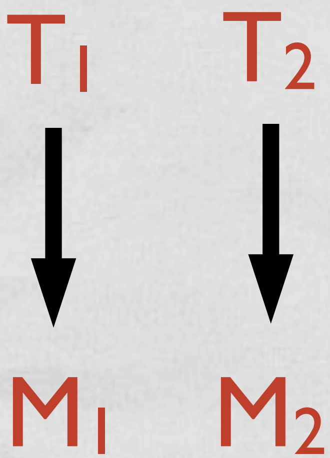
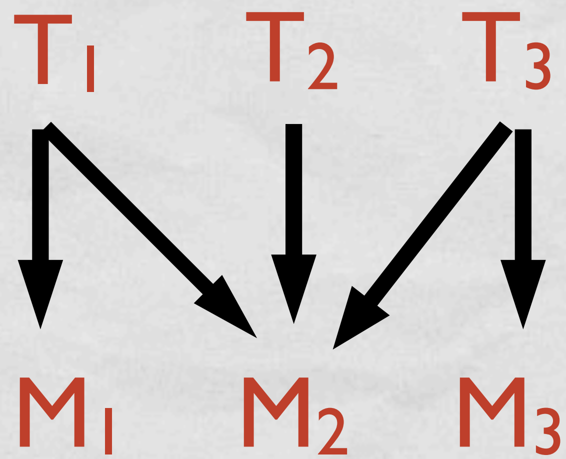
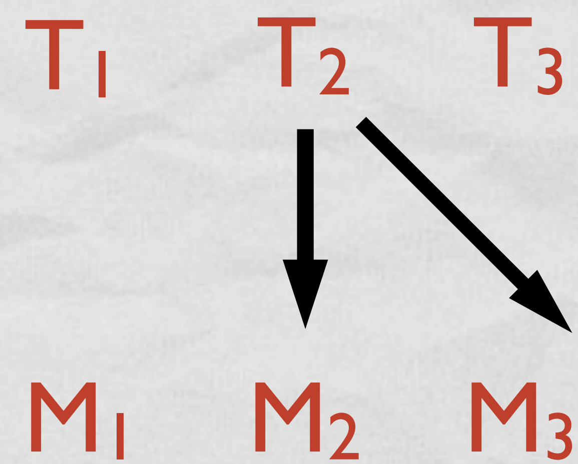






$\exists x \exists y (T(x) \wedge T(y) \wedge \forall z (M(z) \rightarrow (A(x,z) \vee A(y,z))))$

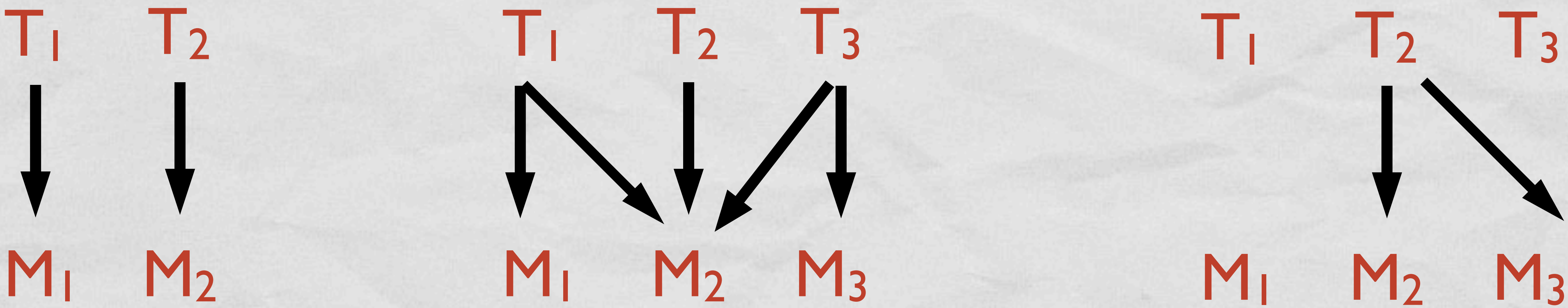
There is a pair of Ts such that for every M, at least one of those Ts went. T, T, F

$\forall x \forall y ((M(x) \wedge M(y)) \rightarrow \exists z (T(z) \wedge (A(z,x) \leftrightarrow A(z,y))))$

For every pair of Ms, there is a T that went to the first if and only if they went to the second. F, T, T

$\forall x (M(x) \rightarrow \exists y \exists z (T(y) \wedge T(z)) \wedge A(y,x) \wedge \neg A(z,x)))$

For every M, there is a pair of Ts such that one went and the other didn't.



$\exists x \exists y(T(x) \land T(y) \land \forall z(M(z) \rightarrow (A(x,z) \lor A(y,z))))$

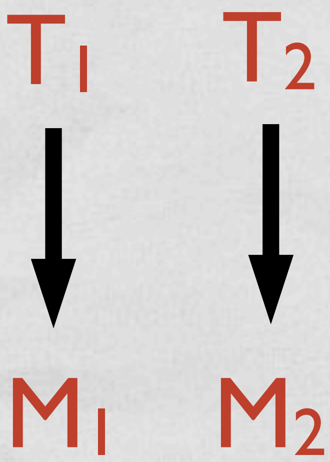
There is a pair of Ts such that for every M, at least one of those Ts went. T, T, F

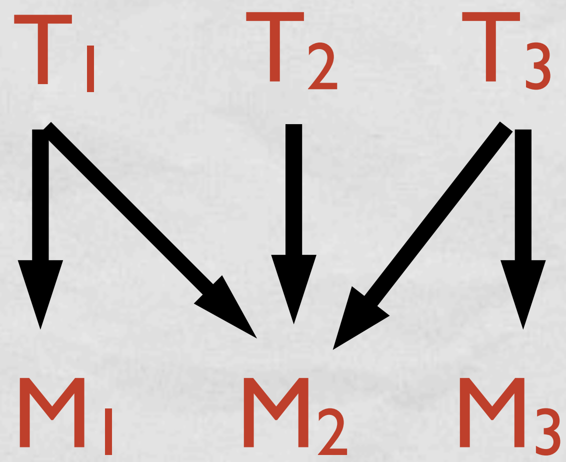
$\forall x \forall y((M(x) \land M(y)) \rightarrow \exists z(T(z) \land (A(z,x) \leftrightarrow A(z,y))))$

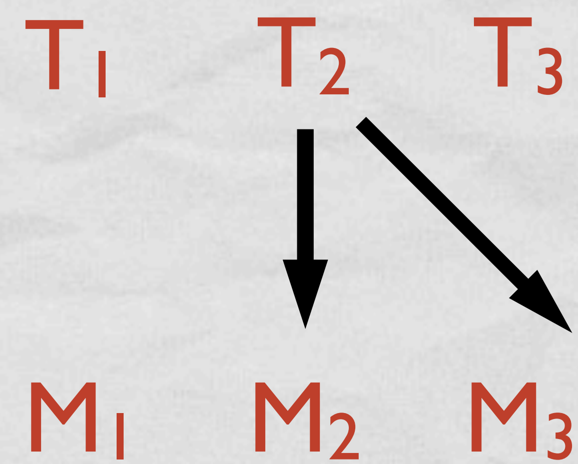
For every pair of Ms, there is a T that went to the first if and only if they went to the second. F, T, T

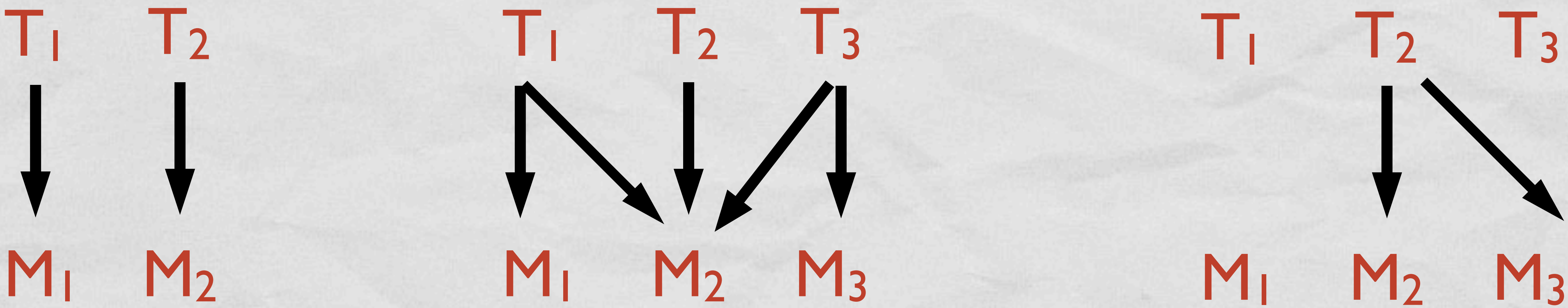
$\forall x(M(x) \rightarrow \exists y \exists z(T(y) \land T(z)) \land A(y,x) \land \neg A(z,x)))$

For every M, there is a pair of Ts such that one went and the other didn't. T, F, F

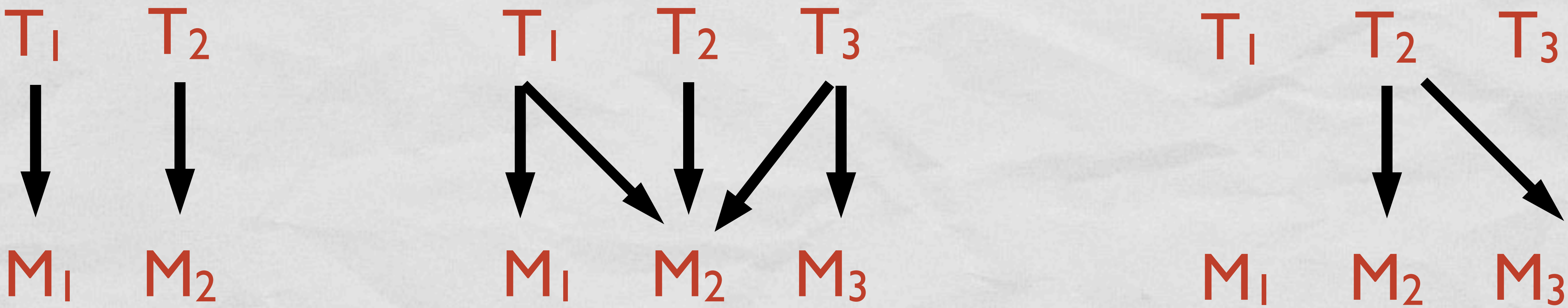

T1
T2
M1
M2


T1
T2
T3
M1
M2
M3


T1
T2
T3
M1
M2
M3

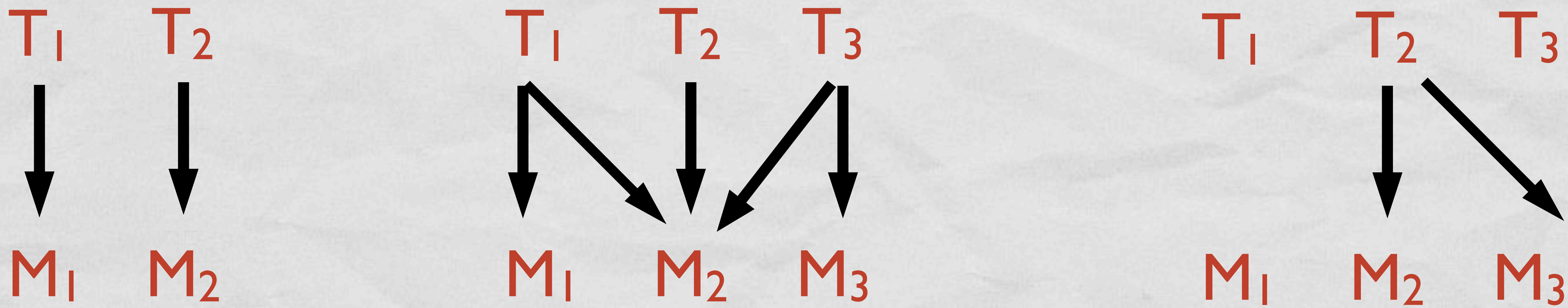


$$\forall x(M(x) \rightarrow \exists y \exists z(T(y) \wedge T(z)) \wedge A(y,x) \leftrightarrow A(z,x)))$$



$\forall x(M(x) \rightarrow \exists y \exists z(T(y) \wedge T(z)) \wedge A(y,x) \leftrightarrow A(z,x)))$

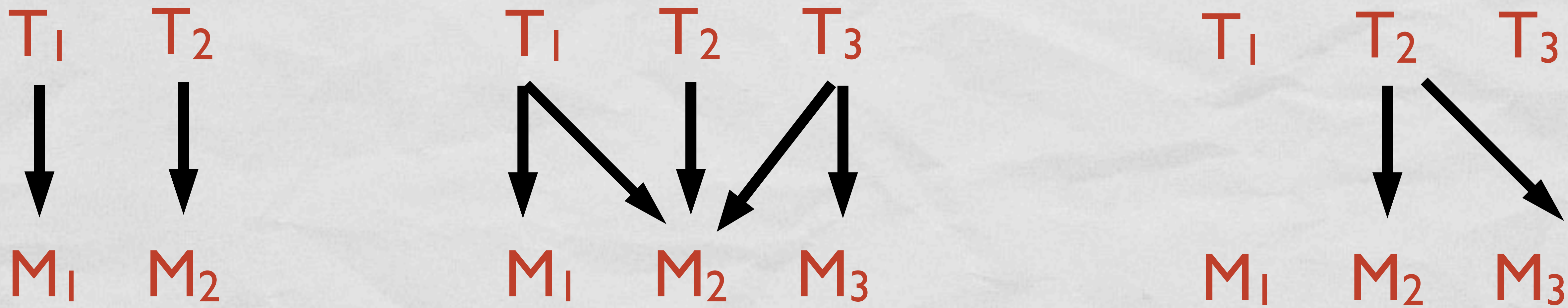
For every M, there is a pair of Ts such that the first went to the M if and only if the second did.



$$\forall x(M(x) \rightarrow \exists y \exists z(T(y) \wedge T(z)) \wedge A(y,x) \leftrightarrow A(z,x)))$$

For every M, there is a pair of Ts such that the first went to the M if and only if the second did.

T, T, T

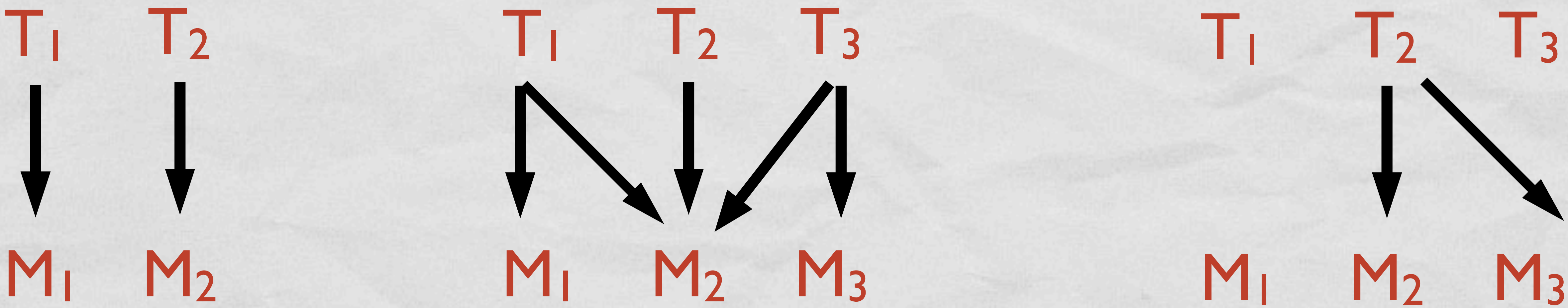


$\forall x(M(x) \rightarrow \exists y \exists z(T(y) \wedge T(z)) \wedge A(y,x) \leftrightarrow A(z,x)))$

For every M, there is a pair of Ts such that the first went to the M if and only if the second did.

T, T, T

How could this be right? What pair could work for say the first diagram? -- Ans, <T_1, T_1>



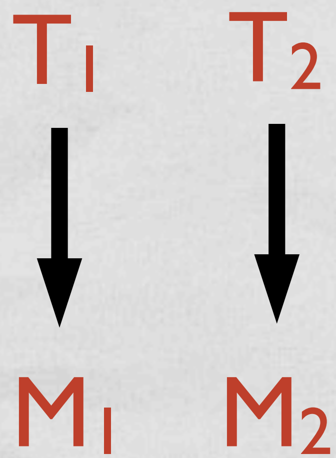
$\forall x(M(x) \rightarrow \exists y \exists z(T(y) \wedge T(z)) \wedge A(y,x) \leftrightarrow A(z,x)))$

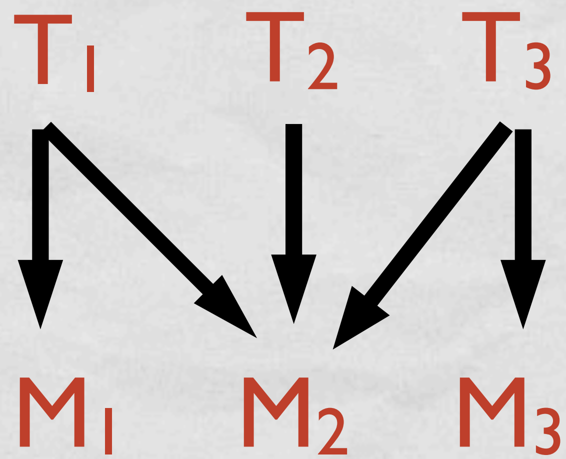
For every M, there is a pair of Ts such that the first went to the M if and only if the second did.

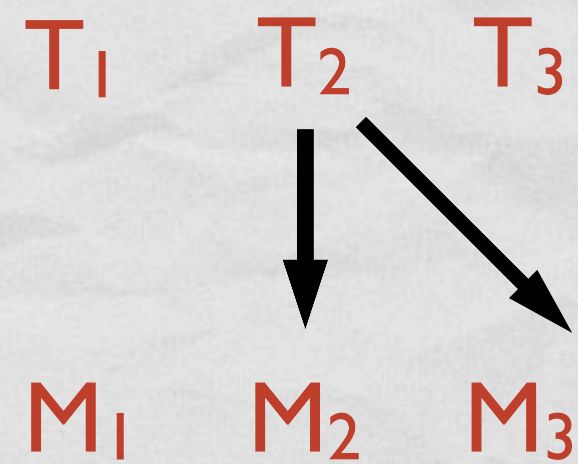
T, T, T

How could this be right? What pair could work for say the first diagram? -- Ans, <T1, T1>

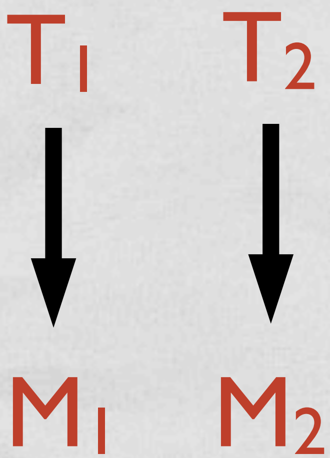
In fact, the above sentence follows just from $\exists z\, T(y)$

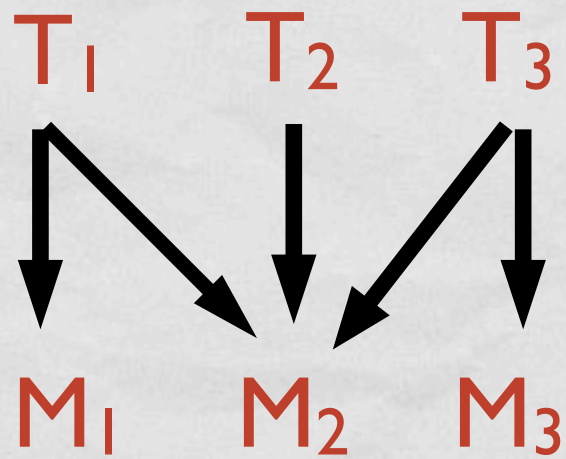


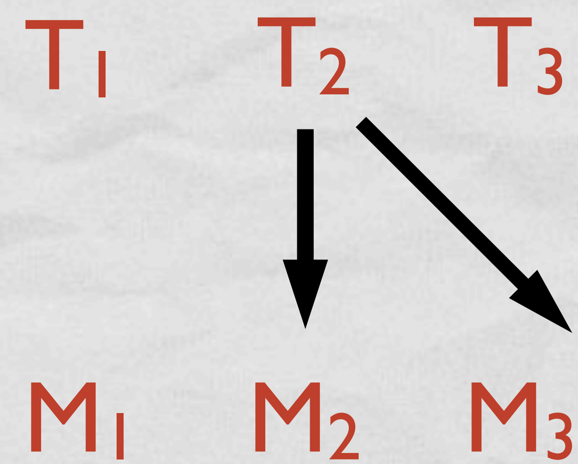




$$\forall x(M(x) \rightarrow \exists y \exists z(T(y) \wedge T(z)) \wedge A(y,x) \leftrightarrow A(z,x)))$$

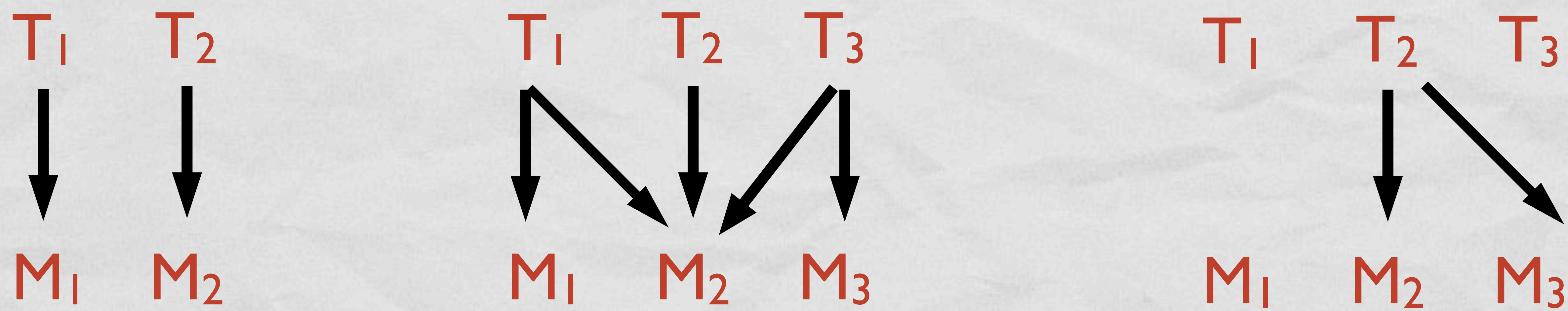






$$\forall x(M(x) \rightarrow \exists y \exists z(T(y) \wedge T(z)) \wedge A(y,x) \leftrightarrow A(z,x)))$$

But sometimes you explicitly want to talk about pairs of distinct Ts - two *different* Ts. For this you need identity.

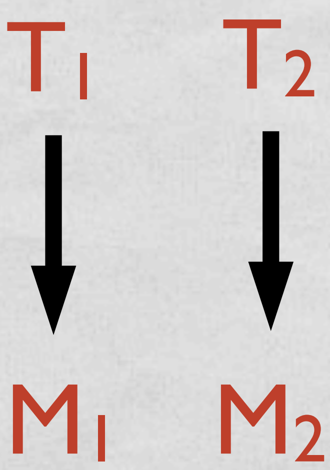


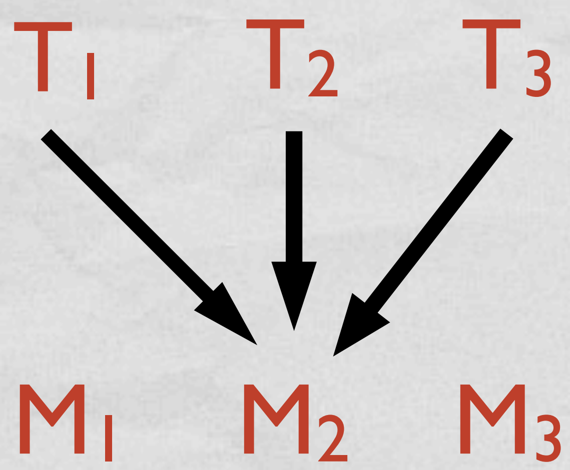
$$\forall x(M(x) \rightarrow \exists y \exists z(T(y) \wedge T(z)) \wedge A(y,x) \leftrightarrow A(z,x)))$$

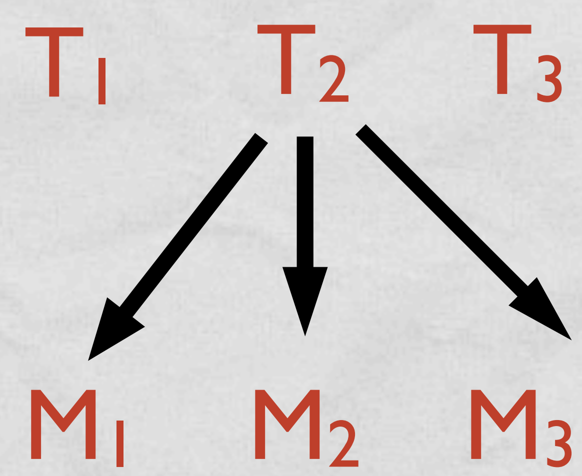
But sometimes you explicitly want to talk about pairs of distinct Ts - two *different* Ts. For this you need identity.

Identity is used whenever you want to *count* things

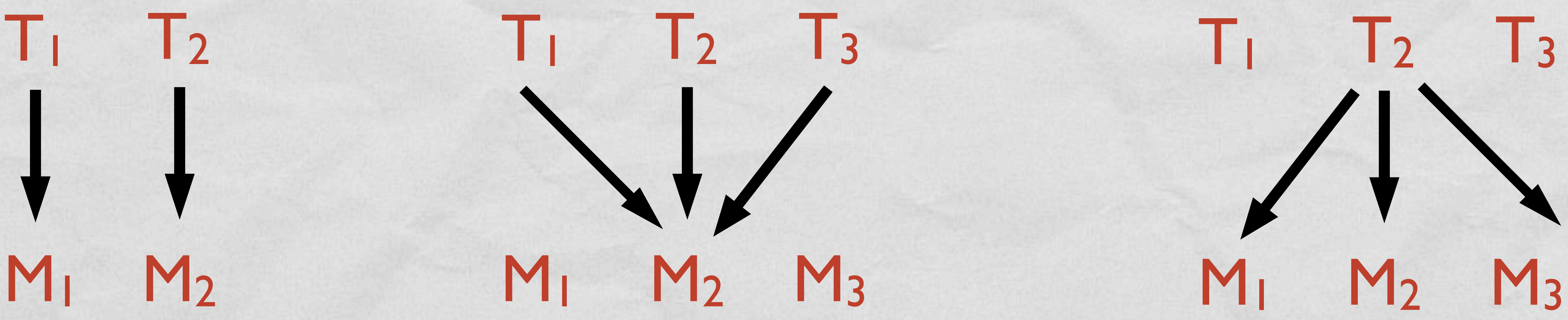
Counting in Diagrams







Counting in Diagrams



A very natural thing you might want to say about these diagrams essentially involves counting. For example, there is one teacher who went to three meetings and two teachers who went to none (true in diagram 3). For this, you need identity.

Translations with Identity

$\exists x \exists y (T(x) \wedge T(y))$

Translations with Identity

$\exists x \exists y(T(x) \wedge T(y))$

Both x and y are teachers

Translations with Identity

$\exists x \exists y(T(x) \land T(y))$

Both x and y are teachers

- but not necessarily different!

Translations with Identity

$\exists x \exists y(T(x) \wedge T(y))$

Both x and y are teachers

- but not necessarily different!

$\exists x \exists y(T(x) \wedge T(y) \wedge x \neq y)$

Translations with Identity

$\exists x \exists y(T(x) \wedge T(y))$

Both x and y are teachers

- but not necessarily different!

$\exists x \exists y(T(x) \wedge T(y) \wedge x \neq y)$

There are at least two teachers

Translations with Identity

$\exists x \exists y(T(x) \wedge T(y))$

Both x and y are teachers

- but not necessarily different!

$\exists x \exists y(T(x) \wedge T(y) \wedge x \neq y)$

There are at least two teachers

$\exists x \exists y(T(x) \wedge T(y) \wedge x \neq y \wedge \forall z(M(z) \rightarrow (A(x,z) \wedge A(y,z)))$

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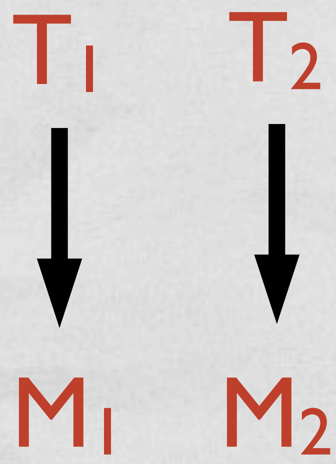
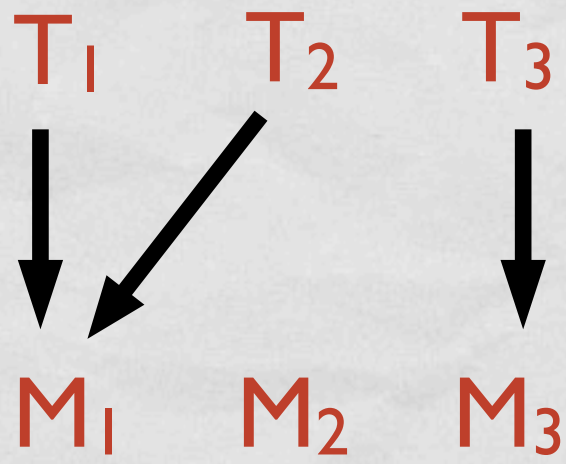
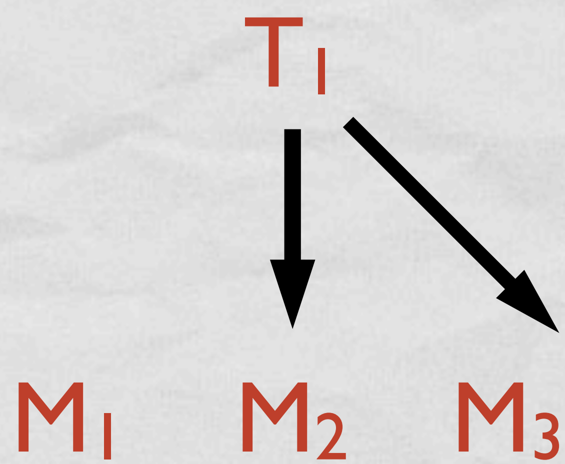
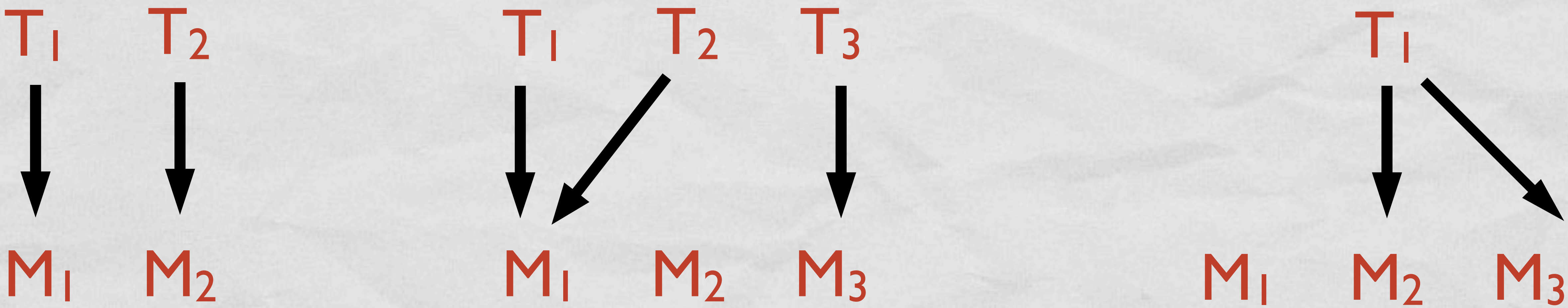
$\exists x \exists y(T(x) \wedge T(y) \wedge x \neq y \wedge \forall z(M(z) \rightarrow (A(x,z) \wedge A(y,z))))$

There are at least two teachers who attended every meeting

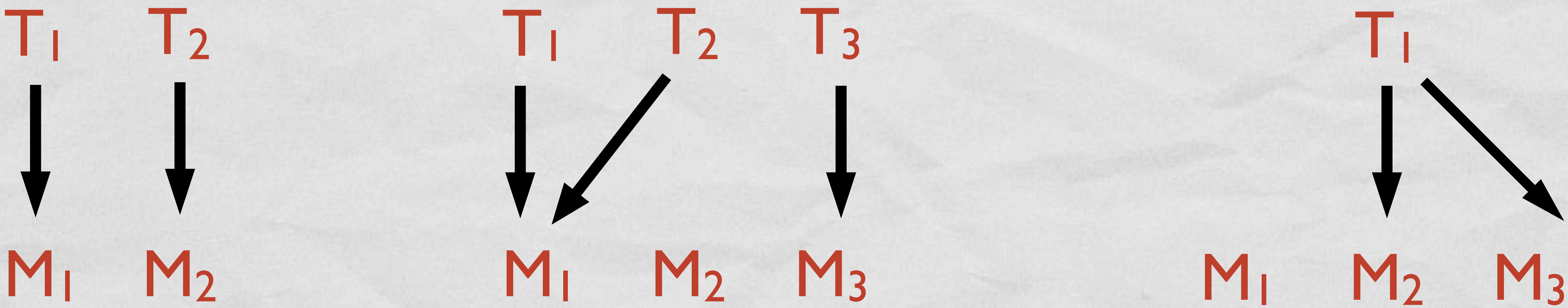

T1
T2
M1
M2


T1
T2
T3
M1
M2
M3


T1
M1
M2
M3

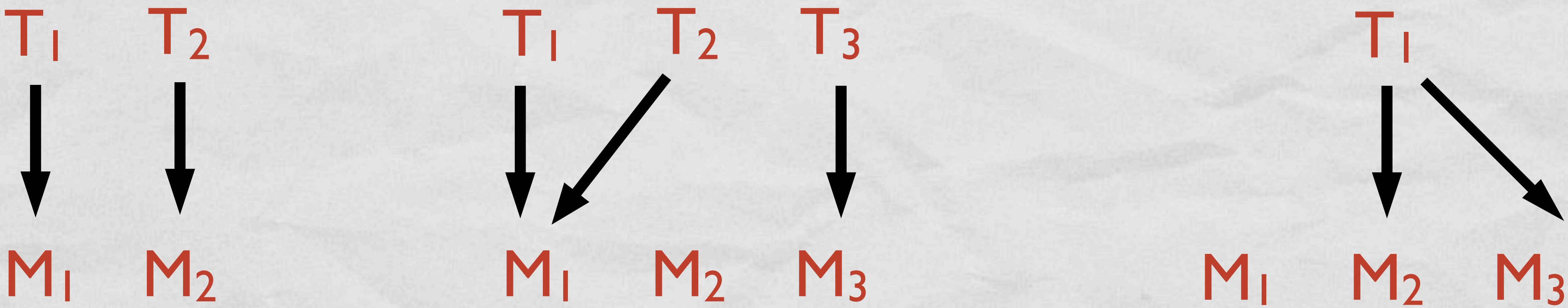


$$\exists x \exists y (T(x) \land T(y) \land x \neq y \land \forall z (M(z) \rightarrow (A(x,z) \leftrightarrow A(y,z))))$$



$$\exists x \exists y (T(x) \wedge T(y) \wedge x \neq y \wedge \forall z (M(z) \rightarrow (A(x,z) \leftrightarrow A(y,z))))$$

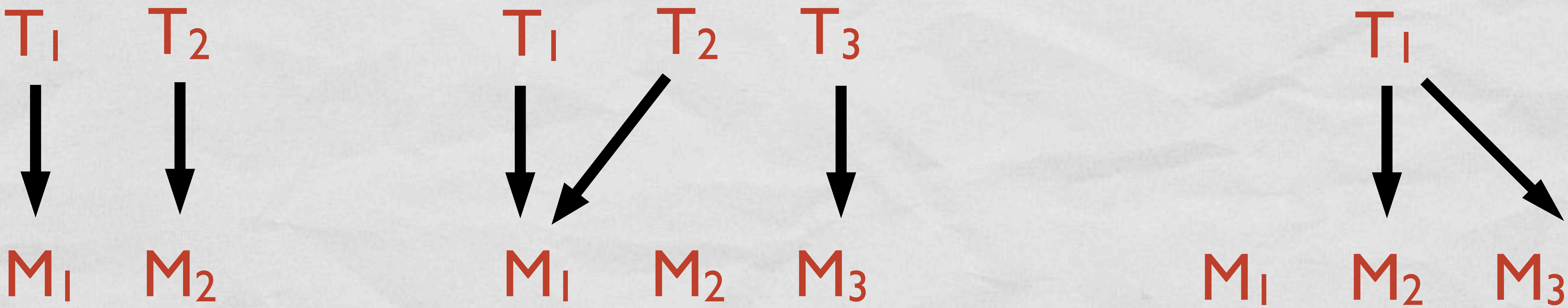
There is a pair of distinct Ts that went to the same Ms



$\exists x \exists y (T(x) \wedge T(y) \wedge x \neq y \wedge \forall z (M(z) \rightarrow (A(x,z) \leftrightarrow A(y,z))))$

There is a pair of distinct Ts that went to the same Ms

F, T, F

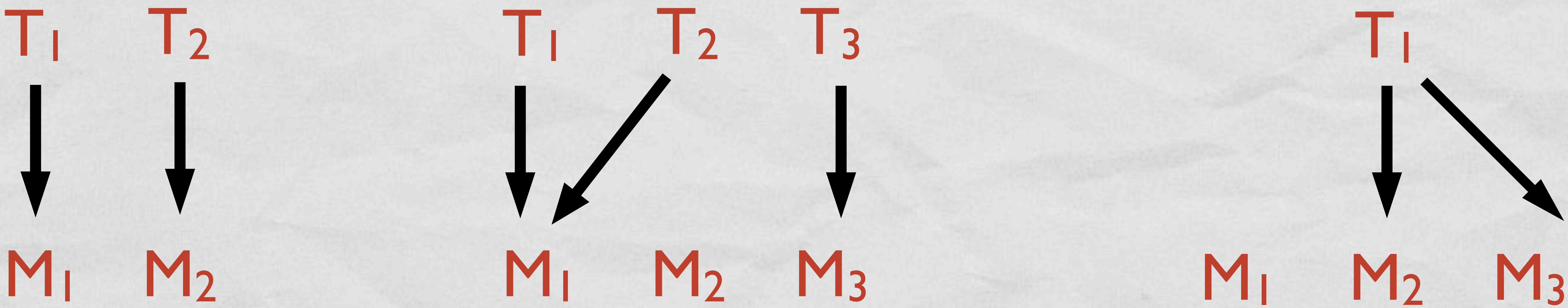


$\exists x \exists y(T(x) \wedge T(y) \wedge x \neq y \wedge \forall z(M(z) \rightarrow (A(x,z) \leftrightarrow A(y,z))))$

There is a pair of distinct Ts that went to the same Ms

F, T, F

$\forall x \forall y((M(x) \wedge M(y) \wedge x \neq y) \rightarrow \exists z(T(z) \wedge A(z,x) \wedge A(z,y))))$



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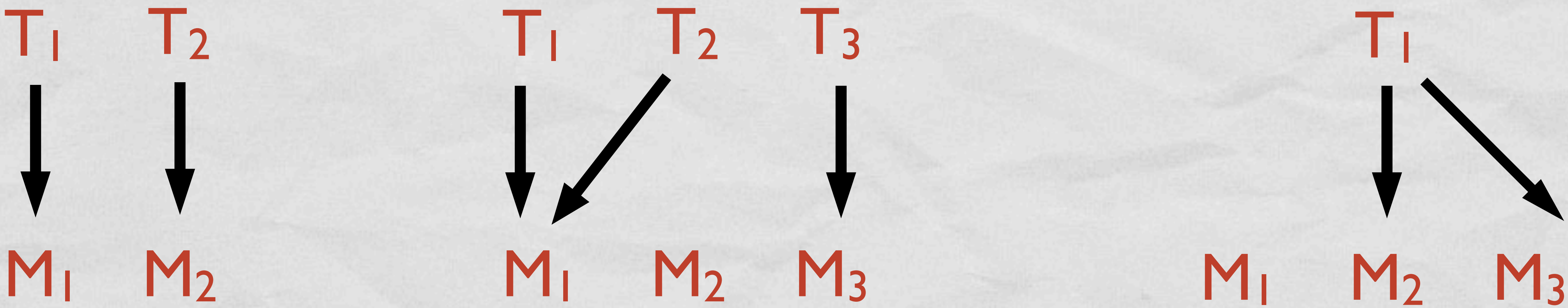
There is a pair of distinct Ts that went to the same Ms

F, T, F

$$\forall x \forall y ((M(x) \wedge M(y) \wedge x \neq y) \rightarrow \exists z (T(z) \wedge A(z,x) \wedge A(z,y)))$$

For every pair of Ms, there is a T that went to both

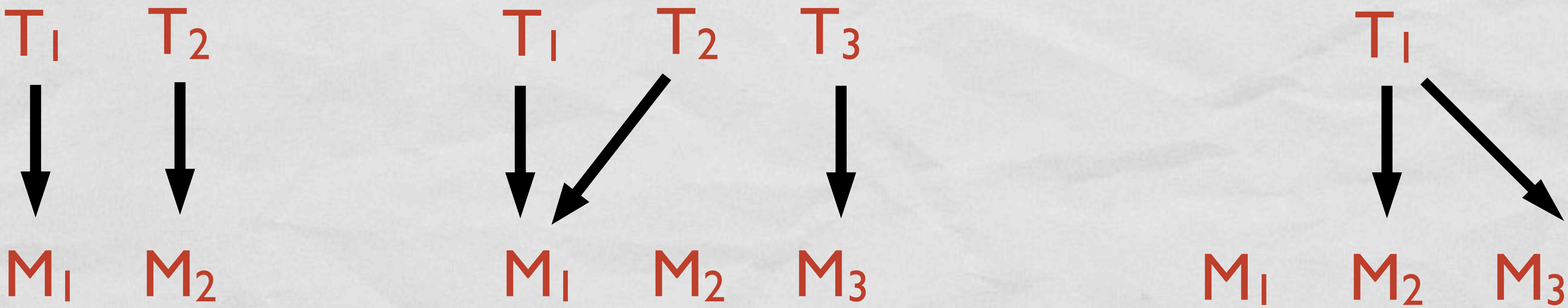


$\exists x \exists y (T(x) \wedge T(y) \wedge x \neq y \wedge \forall z (M(z) \rightarrow (A(x,z) \leftrightarrow A(y,z))))$

There is a pair of distinct Ts that went to the same Ms — F, T, F

$\forall x \forall y ((M(x) \wedge M(y) \wedge x \neq y) \rightarrow \exists z (T(z) \wedge A(z,x) \wedge A(z,y)))$

For every pair of Ms, there is a T that went to both — F, F, F



$\exists x \exists y (T(x) \wedge T(y) \wedge x \neq y \wedge \forall z (M(z) \rightarrow (A(x,z) \leftrightarrow A(y,z))))$

There is a pair of distinct Ts that went to the same Ms

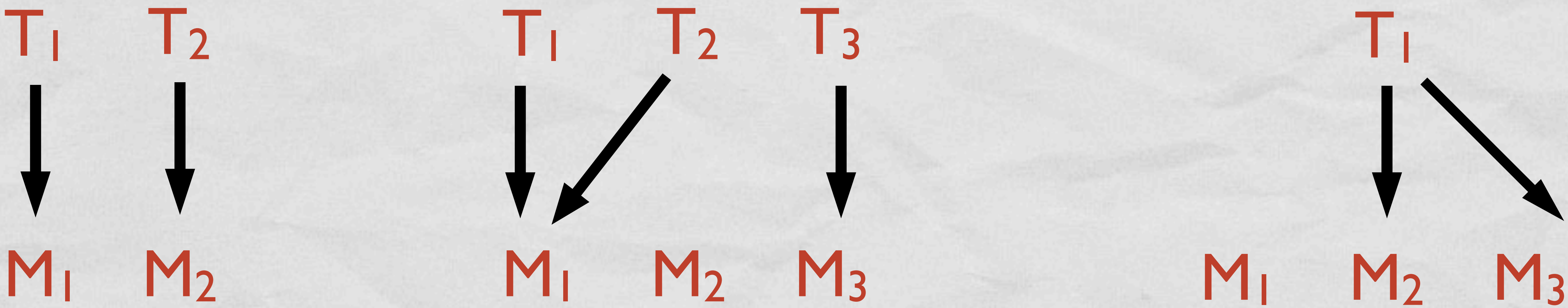
F, T, F

$\forall x \forall y ((M(x) \wedge M(y) \wedge x \neq y) \rightarrow \exists z (T(z) \wedge A(z,x) \wedge A(z,y))))$

For every pair of Ms, there is a T that went to both

F, F, F

$\exists x (T(x) \wedge \exists y \exists z (y \neq z \wedge M(y) \wedge M(z) \wedge A(x,y) \wedge A(x,z))$



$\exists x \exists y(T(x) \wedge T(y) \wedge x \neq y \wedge \forall z(M(z) \rightarrow (A(x,z) \leftrightarrow A(y,z))))$

There is a pair of distinct Ts that went to the same Ms

F, T, F

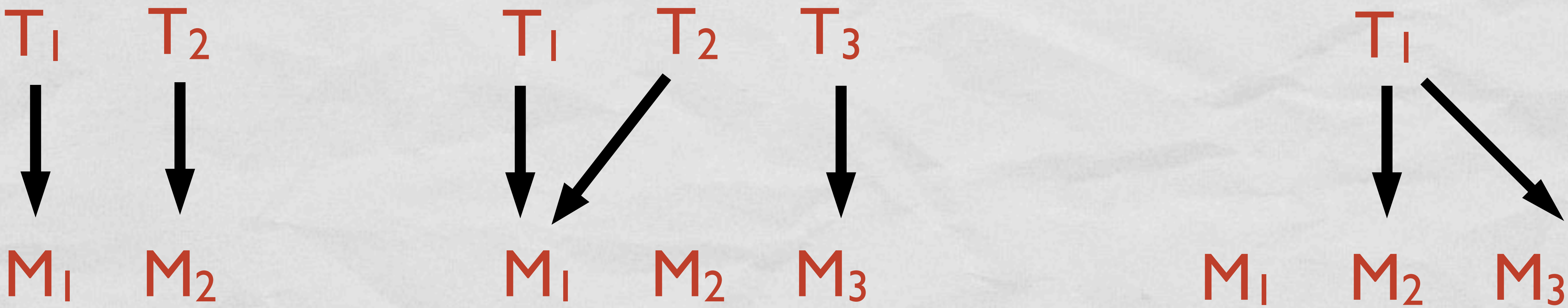
$\forall x \forall y((M(x) \wedge M(y) \wedge x \neq y) \rightarrow \exists z(T(z) \wedge A(z,x) \wedge A(z,y))))$

For every pair of Ms, there is a T that went to both

F, F, F

$\exists x(T(x) \wedge \exists y \exists z(y \neq z \wedge M(y) \wedge M(z) \wedge A(x,y) \wedge A(x,z))$

There is a T who went to two different Ms



$\exists x \exists y(T(x) \wedge T(y) \wedge x \neq y \wedge \forall z(M(z) \rightarrow (A(x,z) \leftrightarrow A(y,z))))$

There is a pair of distinct Ts that went to the same Ms — F, T, F

$\forall x \forall y((M(x) \wedge M(y) \wedge x \neq y) \rightarrow \exists z(T(z) \wedge A(z,x) \wedge A(z,y))))$

For every pair of Ms, there is a T that went to both — F, F, F

$\exists x(T(x) \wedge \exists y \exists z(y \neq z \wedge M(y) \wedge M(z) \wedge A(x,y) \wedge A(x,z))$

There is a T who went to two different Ms — F, F, T