Validity in FOL

Monday, 28 April

Informal Semantics

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- Some arguments are truth-functionally valid - they are valid just in virtue of the meaning of the truth-functional connectives (¬, ∧, ∨, →, ↔) - for any such argument, any truth-value assignment that makes all of the premises true will also make the conclusion true.

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- $\mathcal{F}_{\mathrm{T}}$ is a sound and complete set of rules for proving t-f valid arguments

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- Example: $\forall x\ P(x)$ therefore $\exists y\ P(y)$ is not t-f valid but it is really valid. And we can know this in virtue of First Order Logic - it is FO valid.
- FO valid means that every interpretation that makes the premises true also makes the conclusion true.

Tarski's World Interpretations

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- Tarski's world can illustrate some interpretations. [But not all! If Tarski's world can't falsify it, it doesn't mean FO valid]
- If you can make the premises true and conclusion false in a Tarski world, then the argument is *really* invalid. If you can make a "suitable translation" that shows invalidity, it is FO invalid.

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- A picture with one cube and one tet is an interpretation that makes the premise true and the conclusion false.

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- Domain: Objects in the picture on the screen, P(x) = x is a cube.
- A picture with one cube and one tet is an interpretation that makes the premise true and the conclusion false.
- FO validity completely ignores the meaning of the predicates. LPL talks about "non-sense" predicates. I like "P", "Q", "R", etc. Replace predicates with arbitrary letters like this to test FO validity.

Testing Validity using Tarski's World

$\forall x(\text{Cube}(x) \lor \text{Tet}(x))$
$\exists x(\text{Small}(x) \land \neg\text{Cube}(x))$

$\forall x(\text{Tet}(x) \rightarrow \text{Small}(x))$

Valid or not?

Testing Validity using Tarski's World

$\forall x(Cube(x) \lor Tet(x))$

$\exists x(Small(x) \land \neg Cube(x))$

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Valid or not?

Try to make premises true and conclusion false. If you succeed, it is definitely not valid.

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$\forall x(Cube(x) \lor Tet(x))$

$\exists x(Small(x) \land \neg Cube(x))$

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Valid or not?

(It is invalid - a large tet and a small tet show this)

Try to make premises true and conclusion false. If you succeed, it is definitely not valid.

Testing Validity using Tarski's World

$\forall x(P(x) \vee Q(x))$

$\forall x(P(x) \rightarrow \neg R(x))$

$\exists x\, R(x)$

$\exists x(Q(x) \wedge S(x))$

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Valid or not?

Lets try Px = x is a cube
Qx = x is a dodec
Rx = x is a tet
Sx = x is small

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Valid or not?

This doesn't work. Premise 1 and premise 3 are inconsistent.

Lets try Px = x is a cube
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Valid or not?

But this works - now we can make the premises true and the conclusion false.

Px = x is a cube

Qx = x is a dodec

Rx = x is large

Sx = x is small

Testing Validity using Tarski's World

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$\exists x(P(x) \wedge Q(x) \wedge S(x))$

Valid or not?

We can't even make this true in Tarski's world - there aren't enough 1 place predicates (we would need to add 'striped' or something...)

Testing Validity using Tarski's World

SameRow(a,b)

SameRow(b,a)

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Can't make T, F in Tarski's World. But this clearly depends on the meaning of SameRow. S(a,b) therefore S(b,a) is not FO valid.

Testing Validity using Tarski's World

| SameRow(a,b)
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Valid or not?

Can't make T, F in Tarski's World. But this clearly depends on the meaning of SameRow. S(a,b) therefore S(b,a) is not FO valid.

What if "SameRow(x,y)" meant RightOf(x,y)?

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- FO valid means that any *interpretation* that makes all of the premises true also makes the conclusion true.
- By 'gives the meaning' we just mean gives enough information to make sentences true or false.
- We can't rely on always being able to use Tarski's World. We need to examine other ways of depicting interpretations.

Informal Semantics

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- $\exists x\ P(x)$ does not FO entail $\forall x\ P(x)$
- Domain: Natural numbers {0,1,2 .... }, P(x) = x is even
- Alternate interpretation: Domain: All people, P(x) = x is male.

Varieties of Interpretations

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Domain: Natural numbers

P(x): Even numbers

Q(x): Prime numbers

R(x): Odd numbers

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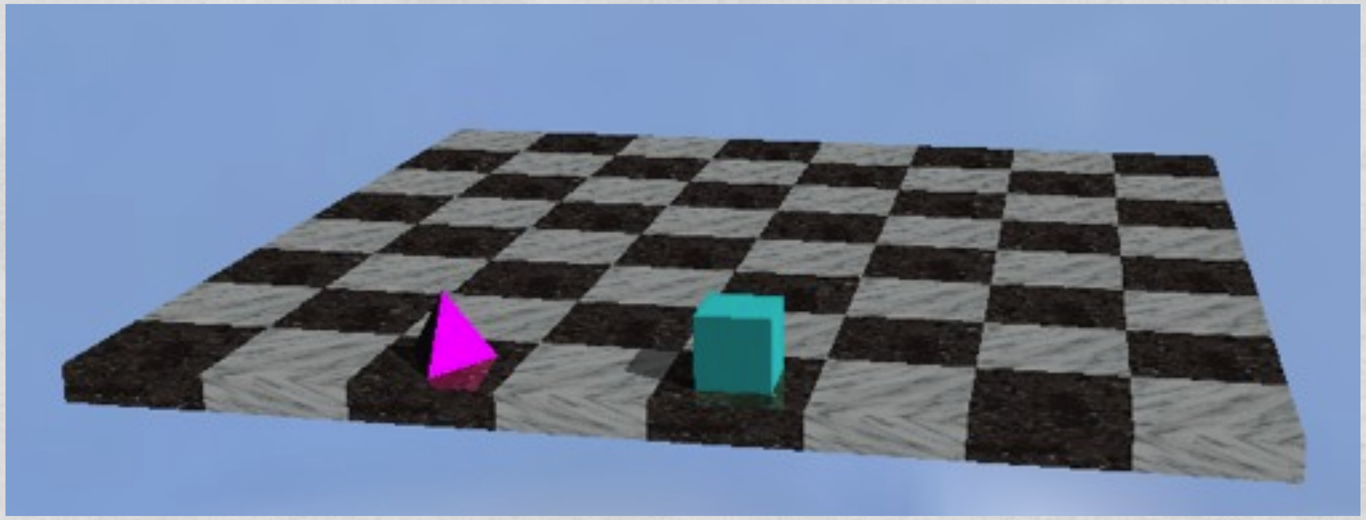
Let P(x)=x is a tet

Q(x) = x is small

R(x) = x is a cube



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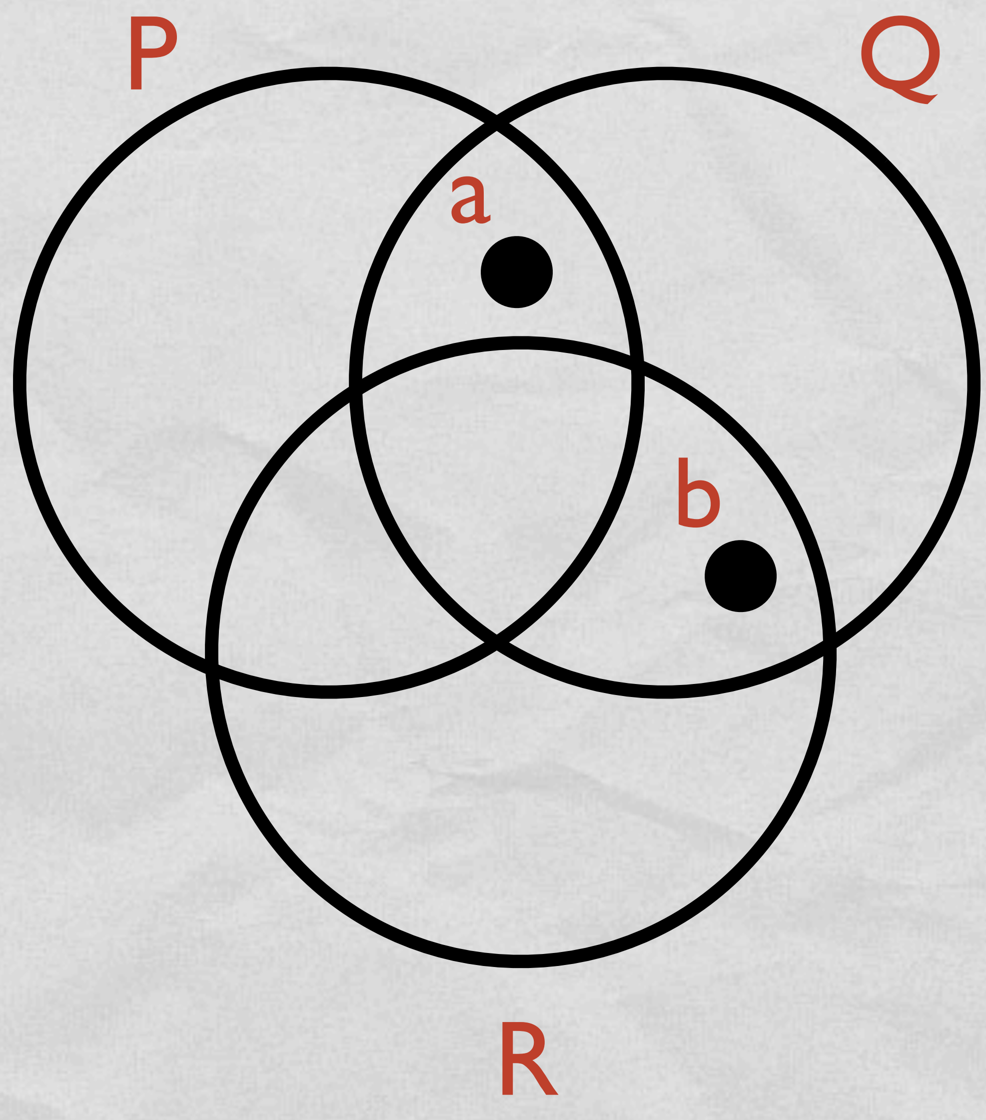

VARIETIES OF INTERPRETATIONS

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Domain: {a,b}

P(x): {a}

Q(x): {a,b}

R(x): {b}

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Q(x): {a,b}

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Called the <u>extension</u> of R

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Domain: dots in my picture.
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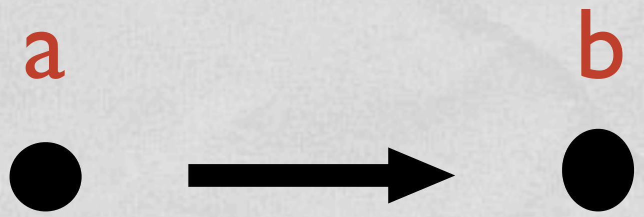

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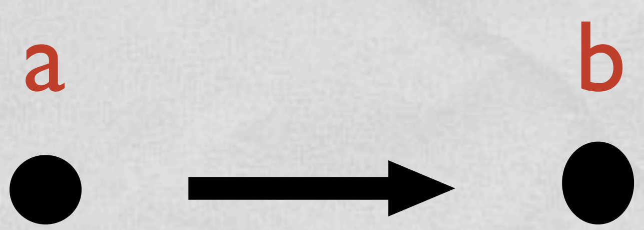

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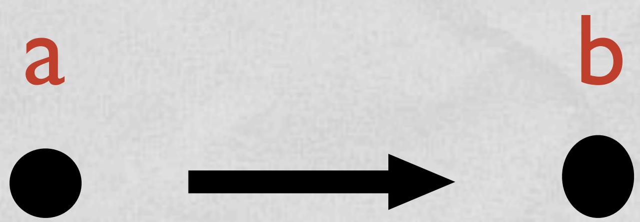

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$\exists x$ R(x,b): True
$\forall x$ R(x,b): False
$\exists x$ R(x,a): False

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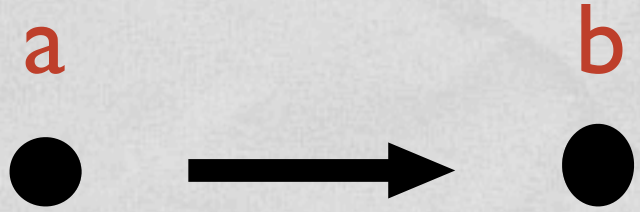

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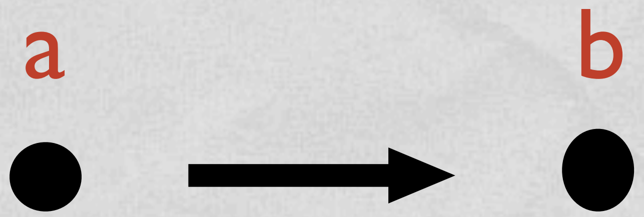

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Of everything, either there is something that it points to, or there is something that points to it

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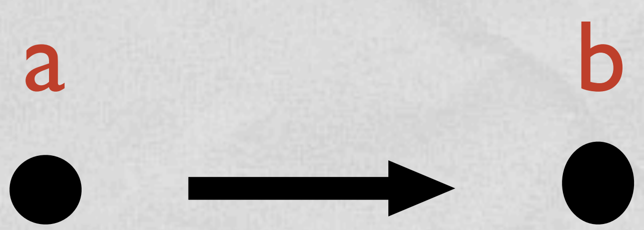

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True

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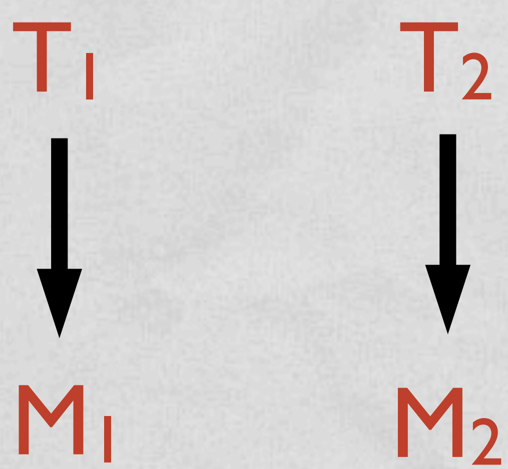
Diagrams

Domain: things in my picture.
A(x,y): x points to y in my picture
T(x): x is labeled 'T' in my picture
M(x): x is labeled 'M' in my picture



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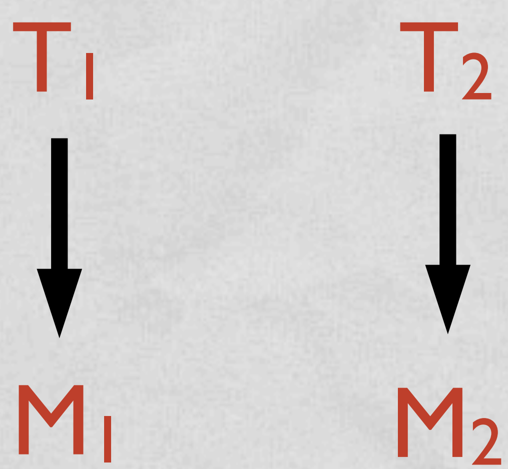

Sometimes it helps to think of English examples. Here teachers attending meetings might be appropriate.

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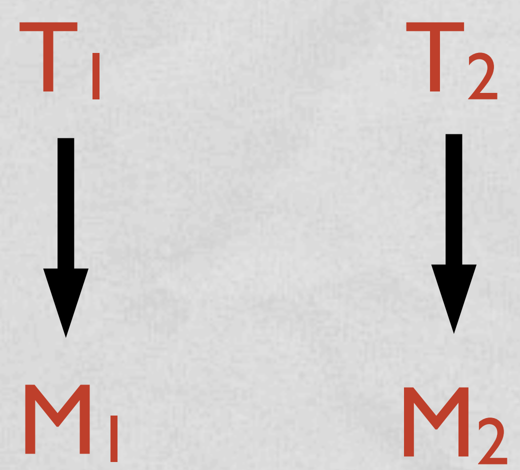

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$\forall x(T(x) \rightarrow \exists y(M(y) \wedge A(x,y)))$ True

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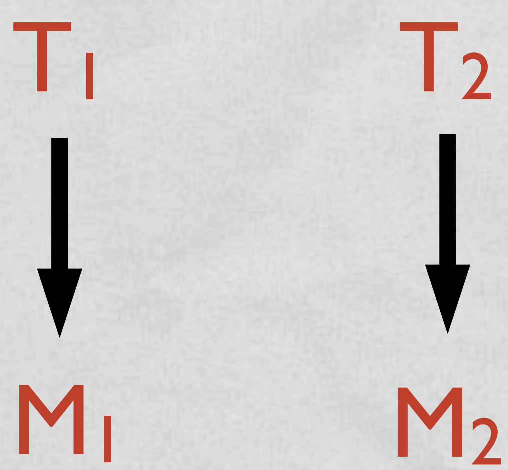

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Mechanical Verification

For ∀x P(x) to be true in an interpretation, P(x) must be satisfied by every element in the domain.

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For $\forall x\, P(x)$ to be true in an interpretation, $P(x)$ must be satisfied by every element in the domain.

In an interpretation with a domain of 3 elements (call them a,b,c), $\forall x\, P(x)$ is true if and only if $P(a) \wedge P(b) \wedge P(c)$ is true. [$\exists x$ with $\vee$]

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$P(a) \wedge P(b) \wedge P(c)$ is true. [$\exists x$ with $\vee$]

$\forall x(P(x) \rightarrow \exists y(Q(y) \wedge R(x,y)))$ is true if and only if

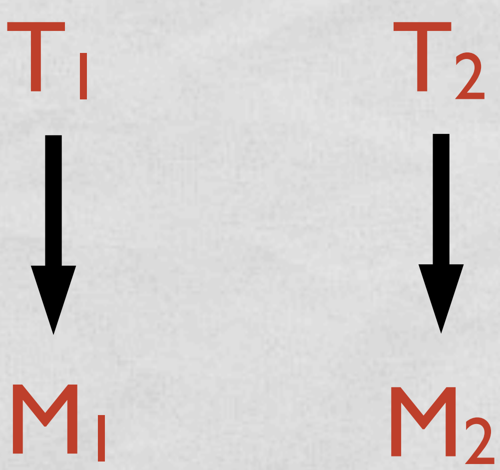
$P(a) \rightarrow \exists y(Q(y) \wedge R(a,y)) \wedge$
$P(b) \rightarrow \exists y(Q(y) \wedge R(b,y)) \wedge$
$P(c) \rightarrow \exists y(Q(y) \wedge R(c,y))$ is true.

Mechanical Verification

T_1 ↓ M_1

T_2 ↓ M_2

Mechanical Verification



$$\forall x(T(x) \rightarrow \exists y(M(y) \wedge A(x,y)))$$

Mechanical Verification

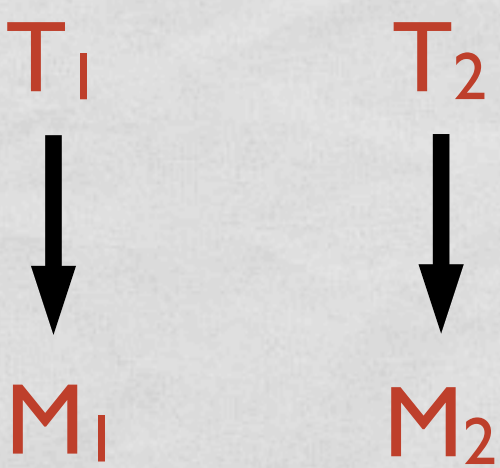
T_1 → M_1

T_2 → M_2

$$\forall x(T(x) \rightarrow \exists y(M(y) \wedge A(x,y)))$$

This universal is true iff a certain conditional is satisfied by all four objects. Since M_1 and M_2 aren't Ts, they satisfy the conditional. So the universal is true just in case the two Ts satisfy it.

Mechanical Verification



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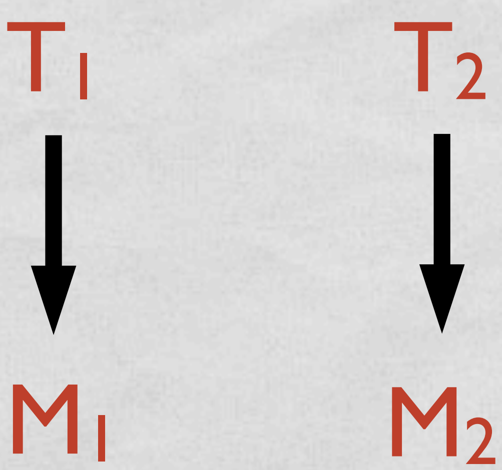
This universal is true iff a certain conditional is satisfied by all four objects. Since M_1 and M_2 aren't Ts, they satisfy the conditional. So the universal is true just in case the two Ts satisfy it.

$T(t_1) \rightarrow \exists y(M(y) \land A(t_1,y))$ True

$T(t_2) \rightarrow \exists y(M(y) \land A(t_2,y))$ True

Mechanical Verification



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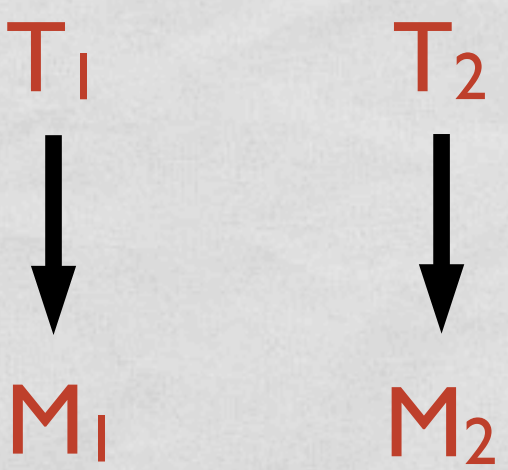
MECHANICAL VERIFICATION



$$\exists x(M(x) \wedge \forall y(T(y) \rightarrow A(y,x)))$$

This existential is true iff a certain conjunction is satisfied by at least one object. Since T_1 and T_2 aren't Ms, they don't satisfy the conjunction. So the existential is true just in case at least one of the two Ms satisfies it.

Mechanical Verification

T_1 T_2

$\downarrow$ $\downarrow$

M_1 M_2

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$M(m_1) \land \forall y(T(y) \rightarrow A(y, m_1))$ False

$M(m_2) \land \forall y(T(y) \rightarrow A(y, m_2))$ False

Mechanical Verification

$T_1 \rightarrow M_1$ $\quad$ $T_2 \rightarrow M_2$

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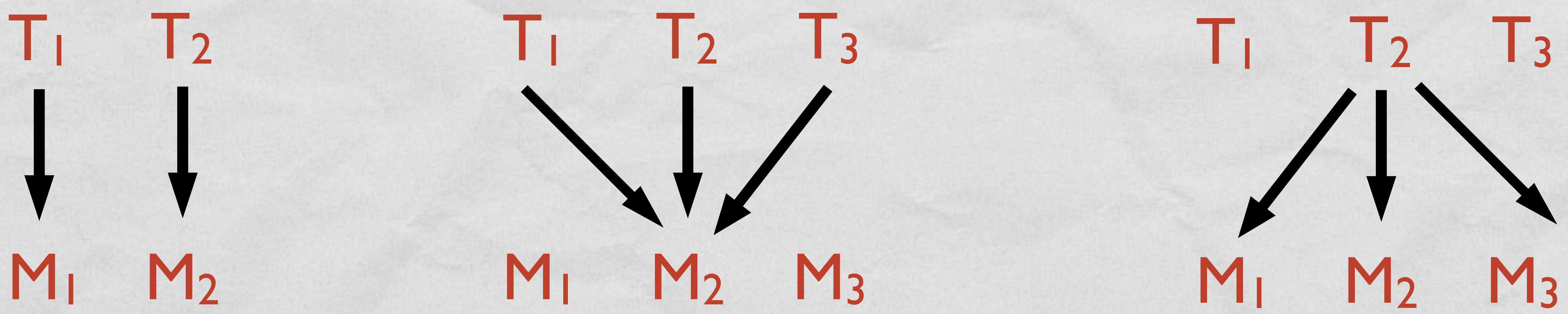
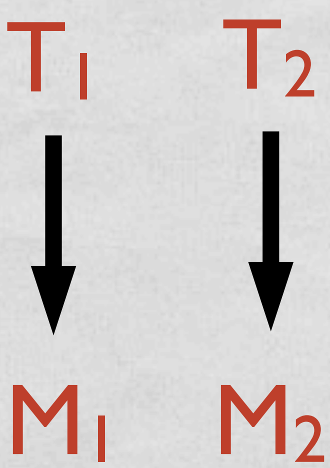
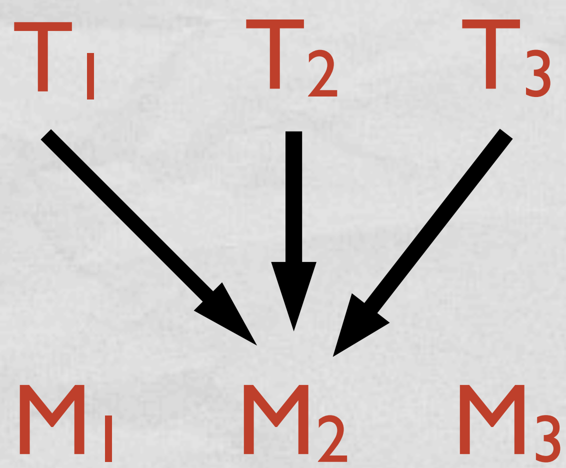
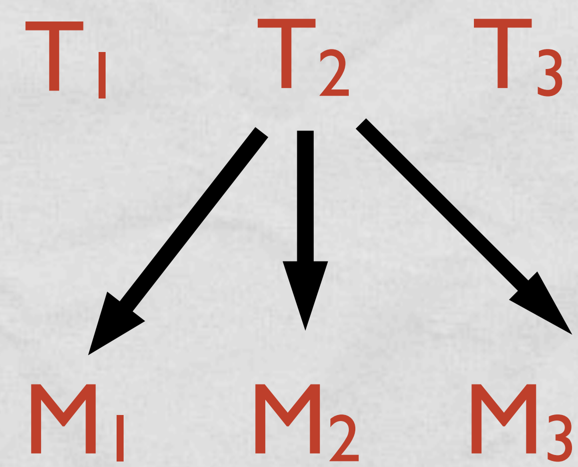
Examples



EXAMPLES







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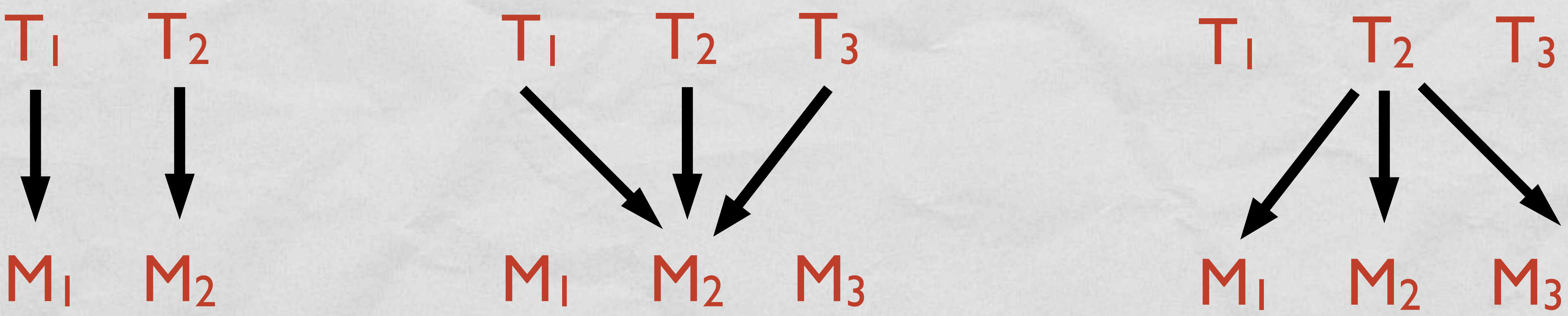

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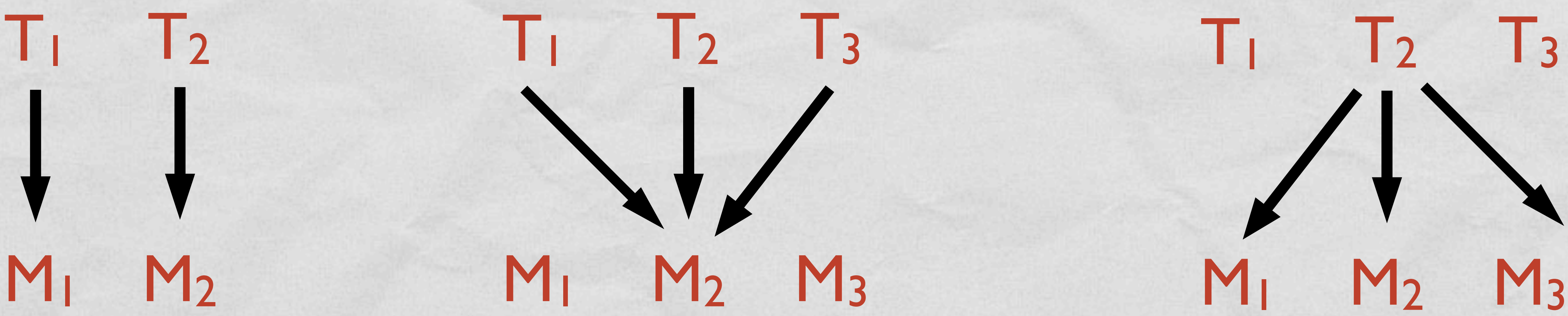

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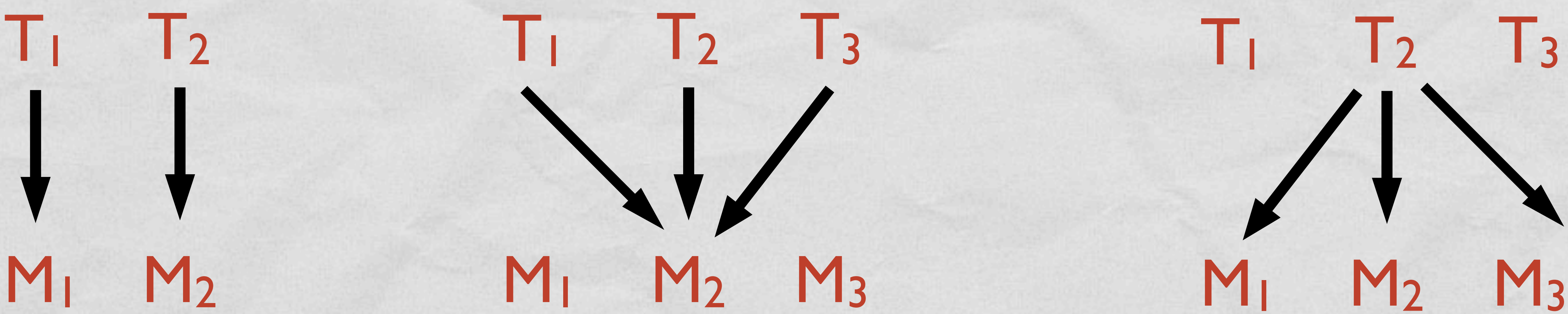
False, True, False

$\forall x(M(x) \rightarrow \exists y(T(y) \wedge A(y,x)))$

Examples



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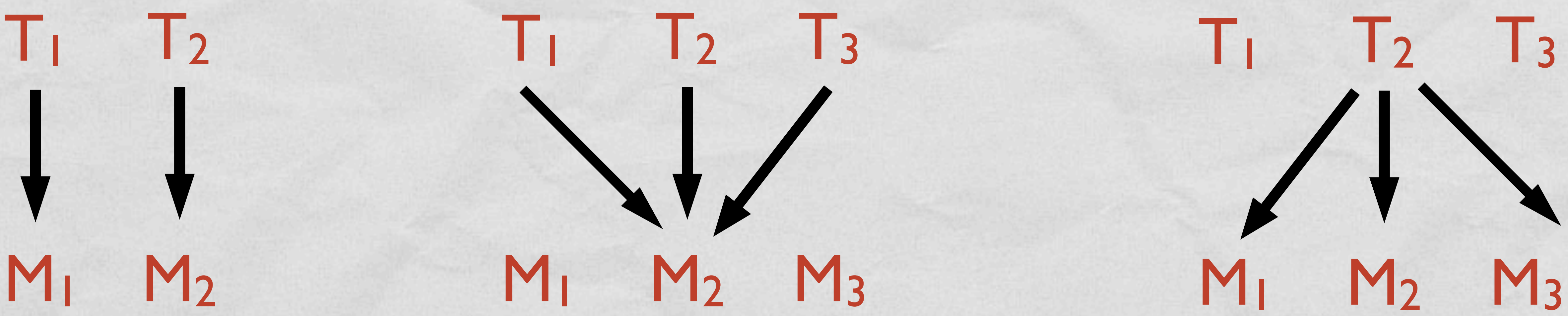

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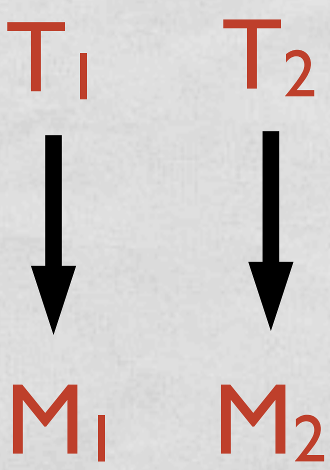
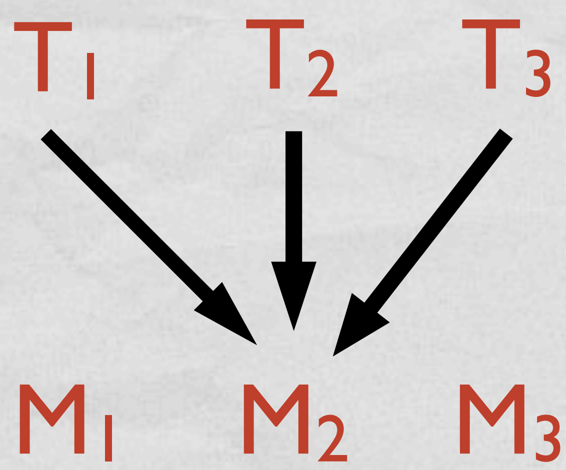
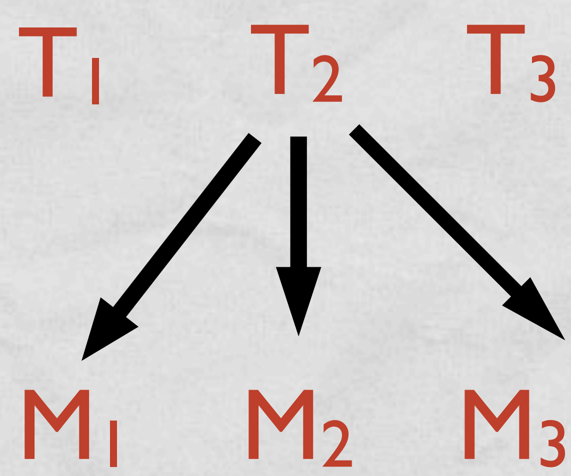
EXAMPLES







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- In addition, with difficult examples, it takes students a lot of effort to come up with an English sentence and it is often wrong or they get the logic wrong because of their sentence.