PUZZLE

You know that at least one (possibly more) of A,B,C are involved in a bank robbery and you know no one else was involved. You also know:

If A is guilty and B is innocent, then C is guilty
C never works alone
A never works with C

Can you safely infer the innocence or guilt of any of them?

Formal Proofs with Existential Quantifiers

Friday, 11 April

Existential Introduction

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- For any variable x, any wff P(x) and any constant c, if we show that P(c), we can conclude that $\exists x\, P(x)$.

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1. P(c)	
2. ∃x P(x)	∃ Intro: 1

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1.	P(c)	
2.	∃x P(x)	∃ Intro: 1

1.	SameRow(b,c)	
2.	∃x SameRow(x,c)	∃ Intro: 1

Existential Introduction

1. Cube(a)
2. ∀x (Small(x) → ¬Cube(x))

∃x ¬Small(x)

Existential Introduction

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---

3. Small(a) → ¬Cube(a) ∀ Elim 2

∃x ¬Small(x)

Existential Introduction

1. Cube(a)
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3. Small(a) → ¬Cube(a) ∀ Elim 2
4. ¬Small(a) Taut Con 1,3

∃x ¬Small(x)

Existential Introduction

1. Cube(a)
2. ∀x (Small(x) → ¬Cube(x))

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3. Small(a) → ¬Cube(a) ∀ Elim 2
4. ¬Small(a) Taut Con 1,3

5. ∃x ¬Small(x) ∃ Intro 5

Existential Elimination

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- If we know that $\exists x\, P(x)$, and we show in a subproof that Q (which does not contain 'c') follows from P(c) where c is arbitrary, we can conclude that Q must be true (outside the subproof).

Existential Elimination

- Existential elimination is like proof by cases ($\vee$ Elim), but with only one case representing an infinite number of cases.
- If we know that $\exists x\, P(x)$, and we show in a subproof that Q (which does not contain 'c') follows from P(c) where c is arbitrary, we can conclude that Q must be true (outside the subproof).
- Note: the constant c must be new. The step will only work if c only occurs within the subproof.

Existential Elimination

1. $\exists x\, P(x)$

2. $\boxed{c}$ P(c)

...

j. Q

7. Q $\exists$ Elim: 1,2-j

Existential Elimination

1. $\exists x$ Cube(x)
2. $\forall x$ (Cube(x) → Large(x))

$\exists x$ Large(x)

Existential Elimination

1. ∃x Cube(x)
2. ∀x (Cube(x) → Large(x))

 3. [a] Cube(a)   (for ∃ Elim)

 ∃x Large(x)

∃x Large(x)   ∃ Elim

Existential Elimination

1. ∃x Cube(x)
2. ∀x (Cube(x) → Large(x))

 3. [a] Cube(a)   (for ∃ Elim)

 4. Cube(a) → Large(a)   ∀ Elim 2

 ∃x Large(x)

∃x Large(x)   ∃ Elim

Existential Elimination

1. ∃x Cube(x)
2. ∀x (Cube(x) → Large(x))

---

3. [a] Cube(a) (for ∃ Elim)

---

4. Cube(a) → Large(a) ∀ Elim 2
5. Large(a) → Elim 3,4

∃x Large(x)

∃x Large(x) ∃ Elim

Existential Elimination

1. ∃x Cube(x)
2. ∀x (Cube(x) → Large(x))

---

3. [a] Cube(a) (for ∃ Elim)

---

4. Cube(a) → Large(a) ∀ Elim 2
5. Large(a) → Elim 3,4

6. ∃x Large(x) ∃ Intro 5

∃x Large(x) ∃ Elim

Existential Elimination

1. ∃x Cube(x)
2. ∀x (Cube(x) → Large(x))

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3. [a] Cube(a) (for ∃ Elim)

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4. Cube(a) → Large(a) ∀ Elim 2
5. Large(a) → Elim 3,4
6. ∃x Large(x) ∃ Intro 5

7. ∃x Large(x) ∃ Elim 1,3-6

Existential Elimination

1. ∃x Cube(x)
2. ∀x (Cube(x) → Large(x))

3. [a] Cube(a) (for ∃ Elim)

4. Cube(a) → Large(a) ∀ Elim 2

5. Large(a) → Elim 3,4

6. ∃x Large(x) ∃ Intro 5

7. ∃x Large(x) ∃ Elim 1,3-6

'a' was totally arbitrary. We could have gotten ∃x Large(x) from Cube(b) or from Cube(c) or ...

Existential Elimination

1. $\exists x(Cube(x) \land RightOf(x,a))$
2. $\forall x\ (RightOf(x,a) \rightarrow SameSize(a,x))$

7. $\exists x(Cube(x) \land SameSize(a,x))$

Existential Elimination

1. ∃x(Cube(x) ∧ RightOf(x,a))
2. ∀x (RightOf(x,a) → SameSize(a,x))

3. [b] Cube(b) ∧ RightOf(b,a) (for ∃ Elim)

6. ∃x(Cube(x) ∧ SameSize(a,x))

7. ∃x(Cube(x) ∧ SameSize(a,x)) ∃ Elim from 1

Existential Elimination

1. ∃x(Cube(x) ∧ RightOf(x,a))
2. ∀x (RightOf(x,a) → SameSize(a,x))

3. b Cube(b) ∧ RightOf(b,a) (for ∃ Elim)

4. RightOf(b,a) → SameSize(a,b) ∀ Elim 2

6. ∃x(Cube(x) ∧ SameSize(a,x))

7. ∃x(Cube(x) ∧ SameSize(a,x)) ∃ Elim from 1

Existential Elimination

1. $\exists x(Cube(x) \wedge RightOf(x,a))$
2. $\forall x\ (RightOf(x,a) \rightarrow SameSize(a,x))$

3. [b] $Cube(b) \wedge RightOf(b,a)$ (for ∃ Elim)

4. $RightOf(b,a) \rightarrow SameSize(a,b)$ ∀ Elim 2

5. $Cube(b) \wedge SameSize(a,b)$ Taut Con 3,4

6. $\exists x(Cube(x) \wedge SameSize(a,x))$

7. $\exists x(Cube(x) \wedge SameSize(a,x))$ ∃ Elim from 1

Existential Elimination

1. $\exists x(\text{Cube}(x) \land \text{RightOf}(x,a))$
2. $\forall x\ (\text{RightOf}(x,a) \rightarrow \text{SameSize}(a,x))$

3. $\boxed{b}$ $\text{Cube}(b) \land \text{RightOf}(b,a)$ (for ∃ Elim)
4. $\text{RightOf}(b,a) \rightarrow \text{SameSize}(a,b)$ ∀ Elim 2
5. $\text{Cube}(b) \land \text{SameSize}(a,b)$ Taut Con 3,4
6. $\exists x(\text{Cube}(x) \land \text{SameSize}(a,x))$ ∃ Intro 5

7. $\exists x(\text{Cube}(x) \land \text{SameSize}(a,x))$ ∃ Elim from 1

Existential Elimination

1. $\exists x(\text{Cube}(x) \land \text{RightOf}(x,a))$
2. $\forall x\ (\text{RightOf}(x,a) \rightarrow \text{SameSize}(a,x))$

---

3. [b] $\text{Cube}(b) \land \text{RightOf}(b,a)$ (for ∃ Elim)

---

4. $\text{RightOf}(b,a) \rightarrow \text{SameSize}(a,b)$ ∀ Elim 2
5. $\text{Cube}(b) \land \text{SameSize}(a,b)$ Taut Con 3,4
6. $\exists x(\text{Cube}(x) \land \text{SameSize}(a,x))$ ∃ Intro 5

7. $\exists x(\text{Cube}(x) \land \text{SameSize}(a,x))$ ∃ Elim 1,3-6

1. $\forall x\ (Tet(x) \rightarrow (SameSize(x,a) \lor SameSize(x,b)))$

2. $\exists x(Tet(x) \land \neg SameSize(x,a))$

$\neg \forall x(SameSize(x,b) \rightarrow SameSize(x,a))$

1. $\forall x\, (Tet(x) \rightarrow (SameSize(x,a) \vee SameSize(x,b)))$

2. $\exists x(Tet(x) \wedge \neg SameSize(x,a))$

3. $\boxed{c}$ $Tet(c) \wedge \neg SameSize(c,a)$ (for ∃ Elim)

$\neg\forall x(SameSize(x,b) \rightarrow SameSize(x,a))$

$\neg\forall x(SameSize(x,b) \rightarrow SameSize(x,a))$ ∃ Elim from 2

1. $\forall x\,(Tet(x) \rightarrow (SameSize(x,a) \vee SameSize(x,b)))$

2. $\exists x(Tet(x) \wedge \neg SameSize(x,a))$

3. [c] $Tet(c) \wedge \neg SameSize(c,a)$ (for ∃ Elim)

4. $Tet(c) \rightarrow (SameSize(c,a) \vee SameSize(c,b))$ ∀ Elim 1

$\neg\forall x(SameSize(x,b) \rightarrow SameSize(x,a))$

$\neg\forall x(SameSize(x,b) \rightarrow SameSize(x,a))$ ∃ Elim from 2

1. $\forall x\, (Tet(x) \rightarrow (SameSize(x,a) \lor SameSize(x,b)))$

2. $\exists x(Tet(x) \land \neg SameSize(x,a))$

3. $\boxed{c}$ $Tet(c) \land \neg SameSize(c,a)$ (for ∃ Elim)

4. $Tet(c) \rightarrow (SameSize(c,a) \lor SameSize(c,b))$ ∀ Elim 1

5. $SameSize(c,b)$ Taut Con 3,4

$\neg \forall x(SameSize(x,b) \rightarrow SameSize(x,a))$

$\neg \forall x(SameSize(x,b) \rightarrow SameSize(x,a))$ ∃ Elim from 2

1. ∀x (Tet(x) → (SameSize(x,a) ∨ SameSize(x,b)))

2. ∃x(Tet(x) ∧ ¬SameSize(x,a))

---

3. [c] Tet(c) ∧ ¬SameSize(c,a) (for ∃ Elim)

---

4. Tet(c) → (SameSize(c,a) ∨ SameSize(c,b)) ∀ Elim 1

5. SameSize(c,b) Taut Con 3,4

6. ∀x(SameSize(x,b) → SameSize(x,a)) (for ¬ Intro)

¬∀x(SameSize(x,b) → SameSize(x,a))

¬∀x(SameSize(x,b) → SameSize(x,a)) ∃ Elim from 2

1. ∀x (Tet(x) → (SameSize(x,a) ∨ SameSize(x,b)))

2. ∃x(Tet(x) ∧ ¬SameSize(x,a))

3. c Tet(c) ∧ ¬SameSize(c,a) (for ∃ Elim)

4. Tet(c) → (SameSize(c,a) ∨ SameSize(c,b)) ∀ Elim 1

5. SameSize(c,b) Taut Con 3,4

6. ∀x(SameSize(x,b) → SameSize(x,a)) (for ¬ Intro)

7. SameSize(c,b) → SameSize(c,a)) ∀ Elim 6

¬∀x(SameSize(x,b) → SameSize(x,a))

¬∀x(SameSize(x,b) → SameSize(x,a)) ∃ Elim from 2

1. $\forall x\,(Tet(x) \rightarrow (SameSize(x,a) \lor SameSize(x,b)))$

2. $\exists x(Tet(x) \land \neg SameSize(x,a))$

---

3. [c] $Tet(c) \land \neg SameSize(c,a)$ (for ∃ Elim)

---

4. $Tet(c) \rightarrow (SameSize(c,a) \lor SameSize(c,b))$ ∀ Elim 1

5. $SameSize(c,b)$ Taut Con 3,4

6. $\forall x(SameSize(x,b) \rightarrow SameSize(x,a))$ (for ¬ Intro)

---

7. $SameSize(c,b) \rightarrow SameSize(c,a))$ ∀ Elim 6

8. $\bot$ Taut Con 3,5,7

$\neg\forall x(SameSize(x,b) \rightarrow SameSize(x,a))$

$\neg\forall x(SameSize(x,b) \rightarrow SameSize(x,a))$ ∃ Elim from 2

1. $\forall x\ (Tet(x) \rightarrow (SameSize(x,a) \lor SameSize(x,b)))$

2. $\exists x(Tet(x) \land \neg SameSize(x,a))$

3. c $Tet(c) \land \neg SameSize(c,a)$ (for ∃ Elim)

4. $Tet(c) \rightarrow (SameSize(c,a) \lor SameSize(c,b))$ ∀ Elim 1

5. $SameSize(c,b)$ Taut Con 3,4

6. $\forall x(SameSize(x,b) \rightarrow SameSize(x,a))$ (for ¬ Intro)

7. $SameSize(c,b) \rightarrow SameSize(c,a))$ ∀ Elim 6

8. $\bot$ Taut Con 3,5,7

9. $\neg\forall x(SameSize(x,b) \rightarrow SameSize(x,a))$ ¬ Intro 6-8

$\neg\forall x(SameSize(x,b) \rightarrow SameSize(x,a))$ ∃ Elim from 2

1. ∀x (Tet(x) → (SameSize(x,a) ∨ SameSize(x,b)))

2. ∃x(Tet(x) ∧ ¬SameSize(x,a))

3. [c] Tet(c) ∧ ¬SameSize(c,a) (for ∃ Elim)

4. Tet(c) → (SameSize(c,a) ∨ SameSize(c,b)) ∀ Elim 1

5. SameSize(c,b) Taut Con 3,4

6. ∀x(SameSize(x,b) → SameSize(x,a)) (for ¬ Intro)

7. SameSize(c,b) → SameSize(c,a)) ∀ Elim 6

8. ⊥ Taut Con 3,5,7

9. ¬∀x(SameSize(x,b) → SameSize(x,a)) ¬ Intro 6-8

10. ¬∀x(SameSize(x,b) → SameSize(x,a)) ∃ Elim 1,3-9