Real LSAT Problem

People with serious financial problems are so worried about money that they cannot be happy. Their misery makes everyone close to them--family, friends colleagues--unhappy as well. Only if their financial problems are solved can they and those around them be happy.

Which one of the following statements can be properly inferred from the passage?

A. Only serious problems make people unhappy.
B. People who solve their serious financial problems will be happy.
C. People who do not have serious financial problems will be happy.
D. If people are unhappy, they have serious financial problems.
E. If people are happy, they do not have serious financial problems.

FORMAL PROOFS WITH QUANTIFIERS

Monday, 7 April

UNIVERSAL ELIMINATION

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- From any sentence of the form $\forall x\ P(x)$, we can infer $P(c)$ (for any constant c).

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1. $\forall x\ P(x)$

2. $P(c)$ ∀ Elim: 1

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1. $\forall x\ P(x)$

2. $P(c)$ $\forall$ Elim: 1

Note: If there are multiple xs in the sentence, replace them all with c

SIMPLE PROOF

1. All men are mortal
2. Socrates is a man
3. Socrates is mortal

SIMPLE PROOF

1. All men are mortal
2. Socrates is a man

---

3. Socrates is mortal

1. $\forall x(Ma(x) \rightarrow Mo(x))$
2. $Ma(s)$

---

3. $Mo(s)$

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---

3. $Ma(s) \rightarrow Mo(s)$ $\forall$Elim 1

SIMPLE PROOF

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3. $Mo(s)$

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2. $Ma(s)$
3. $Ma(s) \rightarrow Mo(s)$ ∀Elim 1
4. $Mo(s)$ →Elim 2,3

UNIVERSAL ELIMINATION

1. $\forall x(\mathrm{Cube}(x) \rightarrow \mathrm{Large}(x))$
2. $\forall x(\mathrm{SameRow}(x,a) \rightarrow \neg\mathrm{Large}(x))$

$\mathrm{SameRow}(b,a) \rightarrow \neg\mathrm{Cube}(b)$

UNIVERSAL ELIMINATION

1. ∀x(Cube(x) → Large(x))
2. ∀x(SameRow(x,a) → ¬Large(x))

3. SameRow(b,a)

¬Cube(b)

SameRow(b,a) → ¬Cube(b) → Intro

UNIVERSAL ELIMINATION

1. ∀x(Cube(x) → Large(x))
2. ∀x(SameRow(x,a) → ¬Large(x))

---

3. SameRow(b,a)

---

4. SameRow(b,a) → ¬Large(b)   ∀Elim 2

¬Cube(b)

SameRow(b,a) → ¬Cube(b)   → Intro

UNIVERSAL ELIMINATION

1. ∀x(Cube(x) → Large(x))
2. ∀x(SameRow(x,a) → ¬Large(x))

3. SameRow(b,a)

4. SameRow(b,a) → ¬Large(b) ∀Elim 2
5. ¬Large(b) → Elim 3,4

¬Cube(b)

SameRow(b,a) → ¬Cube(b) → Intro

UNIVERSAL ELIMINATION

1. ∀x(Cube(x) → Large(x))

2. ∀x(SameRow(x,a) → ¬Large(x))

3. SameRow(b,a)

4. SameRow(b,a) → ¬Large(b) ∀Elim 2

5. ¬Large(b) → Elim 3,4

6. Cube(b) → Large(b) ∀Elim 1

¬Cube(b)

SameRow(b,a) → ¬Cube(b) → Intro

UNIVERSAL ELIMINATION

1. ∀x(Cube(x) → Large(x))

2. ∀x(SameRow(x,a) → ¬Large(x))

---

3. SameRow(b,a)

---

4. SameRow(b,a) → ¬Large(b) ∀Elim 2

5. ¬Large(b) → Elim 3,4

6. Cube(b) → Large(b) ∀Elim 1

7. ¬Cube(b) Taut Con 5,6

8. SameRow(b,a) → ¬Cube(b) → Intro 3-7

UNIVERSAL INTRODUCTION

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- For a constant c naming an arbitrary object, if we show in a subproof that P(c), we can conclude that ∀x P(x).

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- Note: the constant c must be new. The step will only work if c only occurs within the subproof.

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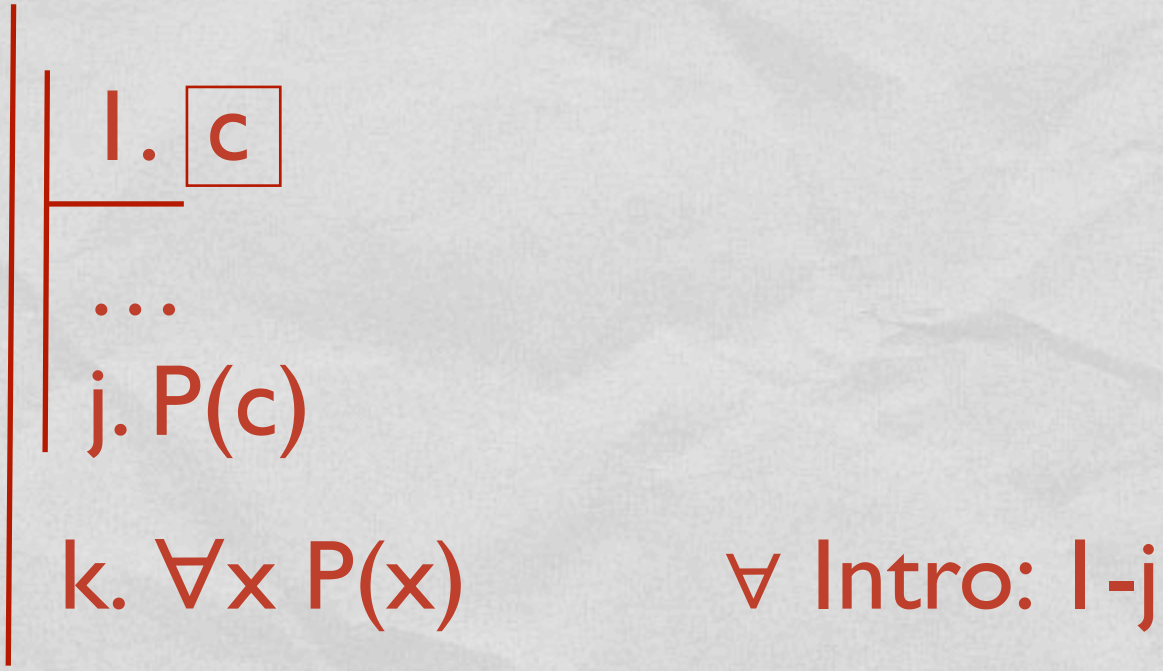
- For a constant c naming an arbitrary object, if we show in a subproof that P(c), we can conclude that ∀x P(x).
- Note: the constant c must be new. The step will only work if c only occurs within the subproof.



1. $\boxed{c}$

…

j. P(c)

k. ∀x P(x) ∀ Intro: 1-j

UNIVERSAL QUANTIFIER PROOFS

1. $\forall x\ \text{Cube}(x)$
2. $\forall x(\text{Cube}(x) \to \text{Large}(x))$

$\forall x\ \text{Large}(x)$

UNIVERSAL QUANTIFIER PROOFS

1. ∀x Cube(x)

2. ∀x(Cube(x) → Large(x))

∀x Large(x) ∀ Intro

UNIVERSAL QUANTIFIER PROOFS

1. ∀x Cube(x)
2. ∀x(Cube(x) → Large(x))

3. $\boxed{a}$

Large(a)

∀x Large(x) ∀ Intro

UNIVERSAL QUANTIFIER PROOFS

1. ∀x Cube(x)
2. ∀x(Cube(x) → Large(x))

> 3. $\boxed{a}$
>
> 4. Cube(a) ∀ Elim 1
>
> Large(a)

∀x Large(x) ∀ Intro

UNIVERSAL QUANTIFIER PROOFS

1. ∀x Cube(x)
2. ∀x(Cube(x) → Large(x))
3. $\boxed{a}$
4. Cube(a) ∀ Elim 1
5. Cube(a) → Large(a) ∀ Elim 2

Large(a)

∀x Large(x) ∀ Intro

UNIVERSAL QUANTIFIER PROOFS

1. ∀x Cube(x)

2. ∀x(Cube(x) → Large(x))

3. $\boxed{a}$

4. Cube(a) ∀ Elim 1

5. Cube(a) → Large(a) ∀ Elim 2

6. Large(a) → Elim 4,5

Large(a)

∀x Large(x) ∀ Intro

UNIVERSAL QUANTIFIER PROOFS

1. ∀x Cube(x)
2. ∀x(Cube(x) → Large(x))

3. $\boxed{a}$
4. Cube(a) ∀ Elim 1
5. Cube(a) → Large(a) ∀ Elim 2
6. Large(a) → Elim 4,5

7. ∀x Large(x) ∀ Intro 3-6

UNIVERSAL QUANTIFIER PROOFS

1. ∀x Cube(x)
2. ∀x(Cube(x) → Large(x))
3. $\boxed{a}$
4. Cube(a) ∀ Elim 1
5. Cube(a) → Large(a) ∀ Elim 2
6. Large(a) → Elim 4,5
7. ∀x Large(x) ∀ Intro 3-6

‘a’ is totally arbitrary. We could have gotten this with any letter. e.g. Large(d)

UNIVERSAL QUANTIFIER PROOFS

1. $\forall x(P(x) \rightarrow Q(x))$
2. $\forall x(Q(x) \rightarrow R(x))$

$\forall x(P(x) \rightarrow R(x))$

UNIVERSAL QUANTIFIER PROOFS

1. $\forall x(P(x) \rightarrow Q(x))$
2. $\forall x(Q(x) \rightarrow R(x))$

$\forall x(P(x) \rightarrow R(x))$ $\forall$ Intro

UNIVERSAL QUANTIFIER PROOFS

1. $\forall x(P(x) \to Q(x))$
2. $\forall x(Q(x) \to R(x))$

3. $\boxed{a}$

$P(a) \to R(a)$

$\forall x(P(x) \to R(x))$ ∀ Intro

UNIVERSAL QUANTIFIER PROOFS

1. $\forall x(P(x) \to Q(x))$
2. $\forall x(Q(x) \to R(x))$

   3. $\boxed{a}$

      4. $P(a)$

         $R(a)$

   $P(a) \to R(a)$ $\to$ Intro

$\forall x(P(x) \to R(x))$ $\forall$ Intro

UNIVERSAL QUANTIFIER PROOFS

1. $\forall x(P(x) \to Q(x))$

2. $\forall x(Q(x) \to R(x))$

3. $\boxed{a}$

4. $P(a)$

5. $P(a) \to Q(a)$ $\forall$ Elim 1

$R(a)$

$P(a) \to R(a)$ $\to$ Intro

$\forall x(P(x) \to R(x))$ $\forall$ Intro

UNIVERSAL QUANTIFIER PROOFS

1. $\forall x(P(x) \to Q(x))$

2. $\forall x(Q(x) \to R(x))$

3. $\boxed{a}$

4. $P(a)$

5. $P(a) \to Q(a)$ ∀ Elim 1

6. $Q(a)$ → Elim 4,5

$R(a)$

$P(a) \to R(a)$ → Intro

$\forall x(P(x) \to R(x))$ ∀ Intro

UNIVERSAL QUANTIFIER PROOFS

1. $\forall x(P(x) \to Q(x))$

2. $\forall x(Q(x) \to R(x))$

3. $\boxed{a}$

4. $P(a)$

5. $P(a) \to Q(a)$ ∀ Elim 1

6. $Q(a)$ → Elim 4,5

7. $Q(a) \to R(a)$ ∀ Elim 2

$R(a)$

$P(a) \to R(a)$ → Intro

$\forall x(P(x) \to R(x))$ ∀ Intro

UNIVERSAL QUANTIFIER PROOFS

1. $\forall x(P(x) \to Q(x))$

2. $\forall x(Q(x) \to R(x))$

3. $\boxed{a}$

4. $P(a)$

5. $P(a) \to Q(a)$ ∀ Elim 1

6. $Q(a)$ → Elim 4,5

7. $Q(a) \to R(a)$ ∀ Elim 2

8. $R(a)$ → Elim 6,7

9. $P(a) \to R(a)$ → Intro 4-8

$\forall x(P(x) \to R(x))$ ∀ Intro

UNIVERSAL QUANTIFIER PROOFS

1. $\forall x(P(x) \rightarrow Q(x))$
2. $\forall x(Q(x) \rightarrow R(x))$

3. $\boxed{a}$

4. $P(a)$

5. $P(a) \rightarrow Q(a)$ ∀ Elim 1

6. $Q(a)$ → Elim 4,5

7. $Q(a) \rightarrow R(a)$ ∀ Elim 2

8. $R(a)$ → Elim 6,7

9. $P(a) \rightarrow R(a)$ → Intro 4-8

$\forall x(P(x) \rightarrow R(x))$ ∀ Intro

‘a’ is totally arbitrary. We could have gotten this with any letter. e.g. $P(j) \rightarrow R(j)$

UNIVERSAL QUANTIFIER PROOFS

1. $\forall x(P(x) \rightarrow Q(x))$
2. $\forall x(Q(x) \rightarrow R(x))$

> 3. $\boxed{a}$
>
> > 4. $P(a)$
> >
> > 5. $P(a) \rightarrow Q(a)$ ∀ Elim 1
> > 6. $Q(a)$ → Elim 4,5
> > 7. $Q(a) \rightarrow R(a)$ ∀ Elim 2
> > 8. $R(a)$ → Elim 6,7
>
> 9. $P(a) \rightarrow R(a)$ → Intro 4-8

10. $\forall x(P(x) \rightarrow R(x))$ ∀ Intro 3-9

1. $\forall x(P(x) \lor Q(x))$
2. $\neg\forall x(S(x) \rightarrow Q(x))$

$\neg\forall x(P(x) \rightarrow \neg S(x))$

1. $\forall x(P(x) \lor Q(x))$

2. $\neg\forall x(S(x) \rightarrow Q(x))$

---

3. $\forall x(P(x) \rightarrow \neg S(x))$ for ¬Intro

$\bot$

$\neg\forall x(P(x) \rightarrow \neg S(x))$ ¬Intro

1. $\forall x(P(x) \vee Q(x))$

2. $\neg\forall x(S(x) \rightarrow Q(x))$

3. $\forall x(P(x) \rightarrow \neg S(x))$ for ¬Intro

$\forall x(S(x) \rightarrow Q(x))$ New goal (to get ⊥ from 2)

$\bot$

$\neg\forall x(P(x) \rightarrow \neg S(x))$ ¬Intro

1. $\forall x(P(x) \vee Q(x))$
2. $\neg\forall x(S(x) \rightarrow Q(x))$

3. $\forall x(P(x) \rightarrow \neg S(x))$ for ¬Intro

4. $\boxed{a}$

$S(a) \rightarrow Q(a)$

$\forall x(S(x) \rightarrow Q(x))$ New goal (to get ⊥ from 2)

$\bot$

$\neg\forall x(P(x) \rightarrow \neg S(x))$ ¬Intro

> 1. $\forall x(P(x) \vee Q(x))$
>
> 2. $\neg\forall x(S(x) \rightarrow Q(x))$
>
> ---
>
> > 3. $\forall x(P(x) \rightarrow \neg S(x))$ for ¬Intro
> >
> > ---
> >
> > > 4. $\boxed{a}$
> > >
> > > ---
> > >
> > > > 5. $S(a)$
> > > >
> > > > ---
> > > >
> > > > $Q(a)$
> > >
> > > $S(a) \rightarrow Q(a)$
> >
> > $\forall x(S(x) \rightarrow Q(x))$ New goal (to get ⊥ from 2)
> >
> > $\bot$
>
> $\neg\forall x(P(x) \rightarrow \neg S(x))$ ¬Intro

> 1. $\forall x(P(x) \vee Q(x))$
>
> 2. $\neg\forall x(S(x) \to Q(x))$
>
> > 3. $\forall x(P(x) \to \neg S(x))$ for ¬Intro
> >
> > > 4. $\boxed{a}$
> > >
> > > > 5. $S(a)$
> > > >
> > > > 6. $P(a) \to \neg S(a)$ ∀ Elim 3
> > > >
> > > > $Q(a)$
> > >
> > > $S(a) \to Q(a)$
> >
> > $\forall x(S(x) \to Q(x))$ New goal (to get $\perp$ from 2)
> >
> > $\perp$
>
> $\neg\forall x(P(x) \to \neg S(x))$ ¬Intro

1. $\forall x(P(x) \lor Q(x))$

2. $\neg\forall x(S(x) \rightarrow Q(x))$

> 3. $\forall x(P(x) \rightarrow \neg S(x))$ for ¬Intro
>
> > 4. $\boxed{a}$
> >
> > > 5. $S(a)$
> > >
> > > 6. $P(a) \rightarrow \neg S(a)$ ∀ Elim 3
> > >
> > > 7. $P(a) \lor Q(a)$ ∀ Elim 1
> > >
> > > $Q(a)$
> >
> > $S(a) \rightarrow Q(a)$
>
> $\forall x(S(x) \rightarrow Q(x))$ New goal (to get ⊥ from 2)
>
> $\perp$

$\neg\forall x(P(x) \rightarrow \neg S(x))$ ¬Intro

1. ∀x(P(x) ∨ Q(x))

2. ¬∀x(S(x) → Q(x))

3. ∀x(P(x) → ¬S(x)) for ¬Intro

4. $\boxed{a}$

5. S(a)

6. P(a) → ¬S(a) ∀ Elim 3

7. P(a) ∨ Q(a) ∀ Elim I

8. Q(a) Taut Con 5,6,7

S(a) → Q(a)

∀x(S(x) → Q(x)) New goal (to get ⊥ from 2)

⊥

¬∀x(P(x) → ¬S(x)) ¬Intro

1. $\forall x(P(x) \vee Q(x))$

2. $\neg\forall x(S(x) \rightarrow Q(x))$

3. $\forall x(P(x) \rightarrow \neg S(x))$ for ¬Intro

4. $\boxed{a}$

5. $S(a)$

6. $P(a) \rightarrow \neg S(a)$ ∀ Elim 3

7. $P(a) \vee Q(a)$ ∀ Elim 1

8. $Q(a)$ Taut Con 5,6,7

9. $S(a) \rightarrow Q(a)$ →Intro 5-8

$\forall x(S(x) \rightarrow Q(x))$ New goal (to get ⊥ from 2)

$\bot$

$\neg\forall x(P(x) \rightarrow \neg S(x))$ ¬Intro

1. $\forall x(P(x) \lor Q(x))$

2. $\neg\forall x(S(x) \rightarrow Q(x))$

> 3. $\forall x(P(x) \rightarrow \neg S(x))$ for ¬Intro
>
> > 4. $\boxed{a}$
> >
> > > 5. $S(a)$
> > >
> > > 6. $P(a) \rightarrow \neg S(a)$ ∀ Elim 3
> > >
> > > 7. $P(a) \lor Q(a)$ ∀ Elim 1
> > >
> > > 8. $Q(a)$ Taut Con 5,6,7
> >
> > 9. $S(a) \rightarrow Q(a)$ →Intro 5-8
>
> 10. $\forall x(S(x) \rightarrow Q(x))$ ∀ Intro 4-9
>
> $\perp$

$\neg\forall x(P(x) \rightarrow \neg S(x))$ ¬Intro

1. $\forall x(P(x) \lor Q(x))$

2. $\neg\forall x(S(x) \rightarrow Q(x))$

> 3. $\forall x(P(x) \rightarrow \neg S(x))$ for ¬Intro
>
> > 4. $\boxed{a}$
> >
> > > 5. $S(a)$
> > >
> > > 6. $P(a) \rightarrow \neg S(a)$ ∀ Elim 3
> > >
> > > 7. $P(a) \lor Q(a)$ ∀ Elim 1
> > >
> > > 8. $Q(a)$ Taut Con 5,6,7
> >
> > 9. $S(a) \rightarrow Q(a)$ →Intro 5-8
>
> 10. $\forall x(S(x) \rightarrow Q(x))$ ∀ Intro 4-9
>
> 11. $\perp$ ⊥ intro 2,10

$\neg\forall x(P(x) \rightarrow \neg S(x))$ ¬Intro

1. $\forall x(P(x) \lor Q(x))$

2. $\neg\forall x(S(x) \rightarrow Q(x))$

3. $\forall x(P(x) \rightarrow \neg S(x))$ for ¬Intro

4. $\boxed{a}$

5. $S(a)$

6. $P(a) \rightarrow \neg S(a)$ ∀ Elim 3

7. $P(a) \lor Q(a)$ ∀ Elim 1

8. $Q(a)$ Taut Con 5,6,7

9. $S(a) \rightarrow Q(a)$ →Intro 5-8

10. $\forall x(S(x) \rightarrow Q(x))$ ∀ Intro 4-9

11. $\perp$ ⊥ intro 2,10

12. $\neg\forall x(P(x) \rightarrow \neg S(x))$ ¬ intro 3-11

1. $\forall x\, P(x) \lor \forall x\, Q(x)$

$\forall x(P(x) \lor Q(x))$

1. $\forall x\, P(x) \lor \forall x\, Q(x)$

$\forall x(P(x) \lor Q(x))$ ∀ Intro

1. $\forall x\, P(x) \lor \forall x\, Q(x)$

2. $\boxed{a}$

$P(a) \lor Q(a)$

$\forall x(P(x) \lor Q(x))$ $\forall$ Intro

1. $\forall x\, P(x) \lor \forall x\, Q(x)$

2. $\boxed{a}$

$P(a) \lor Q(a)$

$\forall x(P(x) \lor Q(x))$ $\quad$ $\forall$ Intro

can’t just plug in ‘a’ for line 1. 1 is not a universal

1. $\forall x\, P(x) \lor \forall x\, Q(x)$

2. $\boxed{a}$

$P(a) \lor Q(a)$

$\forall x (P(x) \lor Q(x))$ ∀ Intro

1. ∀x P(x) ∨ ∀x Q(x)

2. $\boxed{a}$

3. ∀x P(x)

P(a) ∨ Q(a)

∀x Q(x)

P(a) ∨ Q(a)

P(a) ∨ Q(a) ∨ Elim

∀x(P(x) ∨ Q(x)) ∀ Intro

1. $\forall x\, P(x) \lor \forall x\, Q(x)$

2. $\boxed{a}$

3. $\forall x\, P(x)$

4. $P(a)$ $\forall$ Elim 3

$P(a) \lor Q(a)$

$\forall x\, Q(x)$

$Q(a)$

$P(a) \lor Q(a)$

$P(a) \lor Q(a)$ $\lor$ Elim

$\forall x(P(x) \lor Q(x))$ $\forall$ Intro

> 1. $\forall x\, P(x) \lor \forall x\, Q(x)$
>
> > 2. $\boxed{a}$
> >
> > > 3. $\forall x\, P(x)$
> > >
> > > 4. $P(a)$ ∀ Elim 3
> > >
> > > 5. $P(a) \lor Q(a)$ ∨ Intro 4
> >
> > > $\forall x\, Q(x)$
> > >
> > > $Q(a)$
> > >
> > > $P(a) \lor Q(a)$
> >
> > $P(a) \lor Q(a)$ ∨ Elim
>
> $\forall x(P(x) \lor Q(x))$ ∀ Intro

1. $\forall x\, P(x) \vee \forall x\, Q(x)$

2. $\boxed{a}$

3. $\forall x\, P(x)$

4. $P(a)$ $\forall$ Elim 3

5. $P(a) \vee Q(a)$ $\vee$ Intro 4

6. $\forall x\, Q(x)$

7. $Q(a)$ $\forall$ Elim 6

$P(a) \vee Q(a)$

$P(a) \vee Q(a)$ $\vee$ Elim

$\forall x(P(x) \vee Q(x))$ $\forall$ Intro

1. ∀x P(x) ∨ ∀x Q(x)

2. $\boxed{a}$

3. ∀x P(x)

4. P(a) ∀ Elim 3

5. P(a) ∨ Q(a) ∨ Intro 4

6. ∀x Q(x)

7. Q(a) ∀ Elim 6

8. P(a) ∨ Q(a) ∨ Intro 7

9. P(a) ∨ Q(a) ∨ Elim 1,3-5,6-8

10. ∀x(P(x) ∨ Q(x)) ∀ Intro 2-9