PUZZLE

Once I visited the island of knights and knaves, and I met A and B. I asked A "Is either of you a knight?" He responded and after thinking about it, I knew the answer to my question.

What are A and B?

TESTING VALIDITY II

Monday, 10 March

Short Table Method

$P \rightarrow Q$

$\neg Q \vee R$

$R \rightarrow \neg P$

Tautologically Valid or not?

Short Table Method

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$\neg Q \vee R$

$R \rightarrow \neg P$

Tautologically Valid or not?

If not valid, some row of the truth table looks like this:

P	Q	R	$P \rightarrow Q$	$\neg Q \vee R$	$R \rightarrow \neg P$
			T	T	F

Short Table Method

P Q R	$P \rightarrow Q$	$\neg Q \vee R$	$R \rightarrow \neg P$
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Short Table Method

P Q R	$P \rightarrow Q$	$\neg Q \vee R$	$R \rightarrow \neg P$
	T	T	F

Since $R \rightarrow \neg P$ false, R:T, ¬P: F so P:T

Short Table Method

P	Q	R	P → Q	¬Q ∨ R	R → ¬P
T		T	T	T	F

Since R → ¬P false, R:T, ¬P: F so P:T

Short Table Method

P	Q	R	$P \rightarrow Q$	$\neg Q \vee R$	$R \rightarrow \neg P$
T		T	T	T	F

Since $R \rightarrow \neg P$ false, R:T, ¬P: F so P:T

Since $P \rightarrow Q$ true and P, Q:T

Short Table Method

P	Q	R	$P \to Q$	$\neg Q \vee R$	$R \to \neg P$
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Now check to make sure we didn't screw up making $\neg Q \vee R$ True (premise 2)

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Since this row is on the truth table, the argument is **invalid**

Slight different premise 2

P	Q	R	$P \rightarrow Q$	$\neg Q \vee \neg R$	$R \rightarrow \neg P$
			T	T	F

SLIGHT DIFFERENT PREMISE 2

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T		T	T	T	F

Since $R \rightarrow \neg P$ false, R:T, $\neg P$: F so P:T

SLIGHT DIFFERENT PREMISE 2

P	Q	R	$P \to Q$	$\neg Q \lor \neg R$	$R \to \neg P$
T		T	T	T	F

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Now check to make sure we didn't screw up making $\neg Q \vee \neg R$ True (premise 2) - and we did mess up

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Since $P \rightarrow Q$ true and P, Q:T

Now check to make sure we didn't screw up making $\neg Q \vee \neg R$ True (premise 2) - and we did mess up

So there can't be a TTF row on the truth table, so the argument is **valid**

Short Table Method

P	R	S	¬P→(R∧S)	S ↔ P	S ∧ P
			T	T	F

Short Table Method

P	R	S	$\neg P \rightarrow (R \wedge S)$	$S \leftrightarrow P$	$S \wedge P$
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No obvious way to start this - so make a guess. If it works, great. If not, make sure to check the other possibility!

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Since $S \leftrightarrow P$ and $\neg S$, P: F

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Nope - we already made S false. So our assumption (S false) can't lead to a counterexample.

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NOW BACK UP!!

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But S:T and P: F means that premise 2 is false.

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But S:T and P: F means that premise 2 is false.

Since there can't be a counterexample, this argument is valid

Inconsistency and Logical Consequence

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- Notice in all these cases, we are trying to get an assignment where all the premises are true and the conclusion is false. If we do it, the argument is invalid. If this is impossible, then the argument is valid.

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- $\{P_1, P_2, ... P_n, \neg C\}$ is inconsistent (can't all be true at the same time)

  iff

  $\{P_1, P_2, ... P_n\}$ logically entails C

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Provability in a Formal System

- If it is possible to prove C from $\{P_1, P_2, ... P_n\}$ using just the truth functional rules we say that:

$$\{P_1, P_2, ... P_n\} \vdash (\text{in } \mathcal{F}_T)\ C$$

Provability in a Formal System

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We also say that $P_1, P_2, \ldots. P_n \vdash C$ is a valid sequent

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SOUNDNESS THEOREM (for $\mathcal{F}_T$):

If $\{P_1, P_2, ... P_n\} \vdash$(in $\mathcal{F}_T$) C then
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<u>Think Contrapositively</u>

If $\{P_1, P_2, \ldots. P_n\}$ DOES NOT tf-entail C then
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Therefore a falsifying assignment
shows that you can't do a proof

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- For some formal proof system to be sound, it means that anything you can prove in that system really is a valid argument.
- P→Q, Q therefore P really is invalid, but how can I be so sure that I can't prove this in $\mathcal{F}_T$? What if I were really clever?
- I need to show that some relevant fact is true about every one of the infinite number of possible proofs in $\mathcal{F}_T$. Obviously, "check them all" is not the answer.

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[Here I will follow the sketch in section 8.3 in the book]

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[Here I will follow the sketch in section 8.3 in the book]

This is a conditional. I will assume the antecedent (we can do a proof) and try to prove the consequent (the conclusion really does follow from the premises). One way to prove this is to prove the apparently stronger claim that of every step in every line of every proof, the sentence on that line is a consequence of the assumptions “in force” at that line. If that is true of every line, it is true of the last line and so the conclusion would follow from the premises since they are the only assumptions in force.

PROOFS AND COUNTEREXAMPLES

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- You can't both do a proof and find a counterexample. But maybe you could do neither? Is it possible that there is an argument that you can't produce a counterexample for and that you can't do a proof for?
- NO if the argument is t-f valid. (Yes/(maybe?) in general)

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Completeness Theorem

- As a matter of fact, the converse of soundness of $\mathcal{F}_T$ is true - if an argument is tf-valid, then you can do a proof in $\mathcal{F}_T$.
- This is much harder to prove [take 3310 or read chapter 17]. But you can just assume it is true.
- Since $\mathcal{F}_T$ is sound and complete, you can prove all and only the tf-valid arguments. Many other systems of natural deduction have this same quality.