PUZZLE

On the island of knights and knaves, you meet A, B, and C.

A says “Either I am a knight or C is a knave”

B says “I am a knight if and only if C is a knave”

C says “B is a knave”.

Who is what?

The Paradoxes of Material Implication

Monday, 3 March

Paradoxes of Material Implication

Example:

$\neg P \vee Q$

$P \rightarrow Q$

PARADOXES OF MATERIAL IMPLICATION

Example:

| $\neg P \vee Q$
|---
| $P \to Q$

1. $\neg P \vee Q$

$P \to Q$

Paradoxes of Material Implication

Example:

$\neg P \vee Q$

$P \rightarrow Q$

1. $\neg P \vee Q$

2. P — for $\rightarrow$Intro

Q

$P \rightarrow Q$ — $\rightarrow$Intro

Paradoxes of Material Implication

Example:

$\neg P \lor Q$

$P \rightarrow Q$

1. $\neg P \lor Q$

2. P for $\rightarrow$Intro

Now disjunctive syllogism

Q

$P \rightarrow Q$ $\rightarrow$Intro

Paradoxes of Material Implication

Example:

$\neg P \vee Q$

$P \rightarrow Q$

1. $\neg P \vee Q$

2. P for $\rightarrow$Intro

Q

$P \rightarrow Q$ $\rightarrow$Intro

Paradoxes of Material Implication

Example:

$\neg P \lor Q$

$P \rightarrow Q$

1. $\neg P \lor Q$

---

2. P — for $\rightarrow$Intro

---

3. $\neg P$ — for $\lor$ Elim

---

4. $\bot$ — $\bot$ Intro 2,3

Q

$P \rightarrow Q$ — $\rightarrow$Intro

Paradoxes of Material Implication

Example:

$\neg P \vee Q$

$P \rightarrow Q$

1. $\neg P \vee Q$
2. P for $\rightarrow$ Intro
3. $\neg P$ for $\vee$ Elim
4. $\bot$ $\bot$ Intro 2,3

Now I could do $\neg$ intro but I don't want to

Q

$P \rightarrow Q$ $\rightarrow$ Intro

Paradoxes of Material Implication

Example:

$\neg P \vee Q$

$P \rightarrow Q$

1. $\neg P \vee Q$
2. P — for $\rightarrow$Intro
3. $\neg P$ — for $\vee$ Elim
4. $\bot$ — $\bot$ Intro 2,3

The $\bot$ elim rule says from $\bot$, infer anything

Q

$P \rightarrow Q$ — $\rightarrow$Intro

Paradoxes of Material Implication

Example:

$\neg P \vee Q$

$P \rightarrow Q$

1. $\neg P \vee Q$
2. P — for $\rightarrow$Intro
3. $\neg P$ — for $\vee$ Elim
4. $\bot$ — $\bot$ Intro 2,3
5. Q — $\bot$ Elim 4

Q

$P \rightarrow Q$ — $\rightarrow$Intro

The $\bot$ elim rule says from $\bot$, infer anything

Paradoxes of Material Implication

Example:

$\neg P \vee Q$	
$P \rightarrow Q$	

1.	$\neg P \vee Q$	
2.	P	for $\rightarrow$Intro
3.	$\neg P$	for $\vee$ Elim
4.	$\bot$	$\bot$ Intro 2,3
5.	Q	$\bot$ Elim 4
6.	Q	for $\vee$ Elim
	Q	
	$P \rightarrow Q$	$\rightarrow$Intro

Paradoxes of Material Implication

Example:

$\neg P \vee Q$

$P \rightarrow Q$

1.	$\neg P \vee Q$	
2.	P	for $\rightarrow$Intro
3.	$\neg P$	for $\vee$ Elim
4.	$\bot$	$\bot$ Intro 2,3
5.	Q	$\bot$ Elim 4
6.	Q	for $\vee$ Elim
7.	Q	v elim 1, 3-5, 6-6
8.	$P \rightarrow Q$	$\rightarrow$Intro

Paradoxes of Material Implication

Example:

¬P ∨ Q

P→Q

1. ¬P ∨ Q
2. P — for →Intro
3. ¬P — for ∨ Elim
4. ⊥ — ⊥ Intro 2,3
5. Q — ⊥ Elim 4
6. Q — for ∨ Elim
7. Q — v elim 1, 3-5, 6-6
8. P→Q — →Intro 2-7

Paradoxes of Material Implication

We just showed:

$$\begin{array}{|l} \neg P \vee Q \\ \hline P \rightarrow Q \end{array}$$

Paradoxes of Material Implication

We just showed:

$$\begin{array}{|l} \neg P \vee Q \\ \hline P \rightarrow Q \end{array}$$

We did this by showing that:

$$\begin{array}{|l} \neg P \\ \hline P \rightarrow Q \end{array} \quad \text{and} \quad \begin{array}{|l} Q \\ \hline P \rightarrow Q \end{array}$$

Paradoxes of Material Implication

We just showed:

$$\frac{\neg P \vee Q}{P \rightarrow Q}$$

We did this by showing that:

$$\frac{\neg P}{P \rightarrow Q} \quad \text{and} \quad \frac{Q}{P \rightarrow Q}$$

1.	$\neg P$	
2.	P	for $\rightarrow$Intro
3.	$\bot$	$\bot$ Intro 1,2
4.	Q	$\bot$ Elim 3
5.	$P \rightarrow Q$	$\rightarrow$Intro 2-4

Paradoxes of Material Implication

We just showed:

$\neg P \vee Q$ ⊢ $P \rightarrow Q$

We did this by showing that:

$\neg P$ ⊢ $P \rightarrow Q$ and Q ⊢ $P \rightarrow Q$

1.	$\neg P$	
2.	P	for $\rightarrow$Intro
3.	$\bot$	$\bot$ Intro 1,2
4.	Q	$\bot$ Elim 3
5.	$P \rightarrow Q$	$\rightarrow$Intro 2-4

1.	Q	
2.	P	for $\rightarrow$Intro
3.	Q	Reit 1
5.	$P \rightarrow Q$	$\rightarrow$Intro 2-3

The Other Direction

Example:

$P \rightarrow Q$

$\neg P \vee Q$

The Other Direction

Example:

$P \rightarrow Q$

$\neg P \vee Q$

1. $P \rightarrow Q$

$\neg P \vee Q$

THE OTHER DIRECTION

Example:

$P \rightarrow Q$

$\neg P \vee Q$

1. $P \rightarrow Q$

2. $\neg(\neg P \vee Q)$ for ¬Intro

$\bot$ $\bot$ Intro

$\neg P \vee Q$ ¬Intro 2-

The Other Direction

Example:

$P \rightarrow Q$

$\neg P \vee Q$

1. $P \rightarrow Q$

2. $\neg(\neg P \vee Q)$ for ¬Intro

3. $\neg P$ for ¬Intro

$\perp$ $\perp$ Intro

$\neg P \vee Q$ ¬Intro 2-

The Other Direction

Example:

$P \rightarrow Q$

$\neg P \vee Q$

1. $P \rightarrow Q$
2. $\neg(\neg P \vee Q)$ for ¬Intro
3. $\neg P$ for ¬Intro
4. $\neg P \vee Q$ vIntro 3

$\perp$ ⊥ Intro

$\neg P \vee Q$ ¬Intro 2-

The Other Direction

Example:

$P \rightarrow Q$

$\neg P \vee Q$

1. $P \rightarrow Q$
2. $\neg(\neg P \vee Q)$ for ¬Intro
3. $\neg P$ for ¬Intro
4. $\neg P \vee Q$ ∨Intro 3
5. $\bot$ ⊥ Intro 2,4

$\bot$ ⊥ Intro

$\neg P \vee Q$ ¬Intro 2-

THE OTHER DIRECTION

Example:

$P \rightarrow Q$

$\neg P \vee Q$

1. $P \rightarrow Q$
2. $\neg(\neg P \vee Q)$ for ¬Intro
3. $\neg P$ for ¬Intro
4. $\neg P \vee Q$ ∨Intro 3
5. $\bot$ ⊥ Intro 2,4
6. P ¬Intro 3-5

$\bot$ ⊥ Intro

$\neg P \vee Q$ ¬Intro 2-

The Other Direction

Example:

| P → Q
|---
| ¬P ∨ Q

1. P → Q
   2. ¬(¬P ∨ Q) for ¬Intro
      3. ¬P for ¬Intro
      4. ¬P ∨ Q ∨Intro 3
      5. ⊥ ⊥ Intro 2,4
   6. P ¬Intro 3-5
   7. Q →Elim 1,6
   ⊥ ⊥ Intro

¬P ∨ Q ¬Intro 2-

The Other Direction

Example:

$P \rightarrow Q$

$\neg P \vee Q$

1. $P \rightarrow Q$
2. $\neg(\neg P \vee Q)$ for ¬Intro
3. $\neg P$ for ¬Intro
4. $\neg P \vee Q$ ∨Intro 3
5. $\bot$ ⊥ Intro 2,4
6. P ¬Intro 3-5
7. Q →Elim 1,6
8. $\neg P \vee Q$ ∨Intro 7

$\bot$ ⊥ Intro

$\neg P \vee Q$ ¬Intro 2-

The Other Direction

Example:

$P \rightarrow Q$
$\neg P \vee Q$

1. $P \rightarrow Q$
2. $\neg(\neg P \vee Q)$ — for ¬Intro
3. $\neg P$ — for ¬Intro
4. $\neg P \vee Q$ — vIntro 3
5. $\bot$ — ⊥ Intro 2,4
6. P — ¬Intro 3-5
7. Q — →Elim 1,6
8. $\neg P \vee Q$ — vIntro 7
9. $\bot$ — ⊥ Intro 2,8
10. $\neg P \vee Q$ — ¬Intro 2-9

Proving Biconditionals

We have now proved:

$$\begin{array}{|l} P \rightarrow Q \\ \hline \neg P \lor Q \end{array}$$

Proving Biconditionals

We have now proved:

$$\dfrac{P \rightarrow Q}{\neg P \vee Q} \quad \text{and} \quad \dfrac{\neg P \vee Q}{P \rightarrow Q}$$

Proving Biconditionals

We have now proved:

$$\begin{array}{|l} P \rightarrow Q \\ \hline \neg P \vee Q \end{array} \quad \text{and} \quad \begin{array}{|l} \neg P \vee Q \\ \hline P \rightarrow Q \end{array}$$

Therefore we could prove:

$$\begin{array}{|l} \\ \hline (P \rightarrow Q) \leftrightarrow (\neg P \vee Q) \end{array}$$

Proving Biconditionals

$(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$

Proving Biconditionals

$(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$

$(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$

Proving Biconditionals

$(P \rightarrow Q) \leftrightarrow (\neg P \lor Q)$

1. $P \rightarrow Q$ for $\leftrightarrow$Intro

$\neg P \lor Q$

$\neg P \lor Q$ for $\leftrightarrow$Intro

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow (\neg P \lor Q)$ $\leftrightarrow$Intro

1. $P \rightarrow Q$

$\neg P \lor Q$

$\neg P \lor Q$

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow (\neg P \lor Q)$

1. $P \to Q$

$\neg P \vee Q$

$\neg P \vee Q$

$P \to Q$

$(P \to Q) \leftrightarrow (\neg P \vee Q)$

1. $P \to Q$
2. $\neg(\neg P \vee Q)$ — for ¬Intro
3. $\neg P$ — for ¬Intro
4. $\neg P \vee Q$ — ∨Intro 3
5. $\bot$ — ⊥ Intro 2,4
6. P — ¬Intro 3-5
7. Q — →Elim 1,6
8. $\neg P \vee Q$ — ∨Intro 7
9. $\bot$ — ⊥ Intro 2,7
10. $\neg P \vee Q$ — ¬Intro 2-9

1. P→Q

¬P ∨ Q

¬P ∨ Q

P→Q

(P → Q)↔(¬P ∨ Q)

1. P → Q		
2. ¬(¬P ∨ Q)	for ¬Intro	
3. ¬P	for ¬Intro	
4. ¬P ∨ Q	∨Intro 3	
5. ⊥	⊥ Intro 2,4	
6. P	¬Intro 3-5	
7. Q	→Elim 1,6	
8. ¬P ∨ Q	∨Intro 7	
9. ⊥	⊥ Intro 2,7	
10. ¬P ∨ Q	¬Intro 2-9	

1. ¬P ∨ Q	
2. P	for →Intro
3. ¬P	for ∨ Elim
4. ⊥	⊥ Intro 2,3
5. Q	⊥ Elim 4
6. Q	for ∨ Elim
7. Q	∨Elim 1,3-5,6-6
8. P→Q	→Intro 2-7

1. P→Q

¬P ∨ Q

¬P ∨ Q

P→Q

1. P → Q
2. ¬(¬P ∨ Q) for ¬Intro
3. ¬P for ¬Intro
4. ¬P ∨ Q ∨Intro 3
5. ⊥ ⊥ Intro 2,4
6. P ¬Intro 3-5
7. Q →Elim 1,6
8. ¬P ∨ Q ∨Intro 7
9. ⊥ ⊥ Intro 2,7
10. ¬P ∨ Q ¬Intro 2-9

1. ¬P ∨ Q
2. P for →Intro
3. ¬P for ∨ Elim
4. ⊥ ⊥ Intro 2,3
5. Q ⊥ Elim 4
6. Q for ∨ Elim
7. Q ∨Elim 1,3-5,6-6
8. P→Q →Intro 2-7

(P → Q)↔(¬P ∨ Q) ↔Intro 1-10, 11-18

Biconditionals and Equivalence

We have now proved:

$$\vdash (P \to Q) \leftrightarrow (\neg P \lor Q)$$

If a biconditional is a logical truth then the two parts are logically equivalent:

$$(P \to Q) \Leftrightarrow (\neg P \lor Q)$$

Chains of Equivalence

$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

Chains of Equivalence

$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

By DeMorgan's

$(\neg P \vee Q) \Leftrightarrow \neg(P \wedge \neg Q)$

Chains of Equivalence

$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

By DeMorgan's

$(\neg P \vee Q) \Leftrightarrow \neg(P \wedge \neg Q)$

Therefore

$(P \rightarrow Q) \Leftrightarrow \neg(P \wedge \neg Q)$

Chains of Equivalence

$(P \rightarrow Q) \Leftrightarrow (\neg P \lor Q)$

By DeMorgan's

$(\neg P \lor Q) \Leftrightarrow \neg(P \land \neg Q)$

Therefore

$(P \rightarrow Q) \Leftrightarrow \neg(P \land \neg Q)$

and so

$\neg(P \rightarrow Q) \Leftrightarrow (P \land \neg Q)$

Chains of Equivalence

$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$ ⟵ When a conditional is true

By DeMorgan's

$(\neg P \vee Q) \Leftrightarrow \neg(P \wedge \neg Q)$

Therefore

$(P \rightarrow Q) \Leftrightarrow \neg(P \wedge \neg Q)$

and so

$\neg(P \rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$

Chains of Equivalence

$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$ ← When a conditional is true

By DeMorgan's

$(\neg P \vee Q) \Leftrightarrow \neg(P \wedge \neg Q)$

Therefore

$(P \rightarrow Q) \Leftrightarrow \neg(P \wedge \neg Q)$

and so

$\neg(P \rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$ ← When a conditional is false

Negated Conditionals

$\vdash (P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$

Negated Conditionals

$(P \rightarrow Q) \leftrightarrow \neg(P \land \neg Q)$

$(P \rightarrow Q) \leftrightarrow \neg(P \land \neg Q)$

Negated Conditionals

$\vdash (P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$

1. $P \rightarrow Q$ for $\leftrightarrow$Intro

$\neg(P \wedge \neg Q)$

$\neg(P \wedge \neg Q)$ for $\leftrightarrow$Intro

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$ $\leftrightarrow$Intro

1. $P \to Q$ for $\leftrightarrow$Intro

$\neg(P \wedge \neg Q)$

$\neg(P \wedge \neg Q)$ for $\leftrightarrow$Intro

$P \to Q$

$(P \to Q) \leftrightarrow \neg(P \wedge \neg Q)$ $\leftrightarrow$Intro

1. $P \to Q$ for ↔Intro

2. $P \land \neg Q$ for ¬ Intro

$\neg(P \land \neg Q)$

$\neg(P \land \neg Q)$ for ↔Intro

$P \to Q$

$(P \to Q) \leftrightarrow \neg(P \land \neg Q)$ ↔Intro

1. P→Q for ↔Intro

2. P∧¬Q for ¬ Intro

3. P ∧Elim2

¬(P ∧ ¬Q)

¬(P ∧ ¬Q) for ↔Intro

P→Q

(P → Q) ↔ ¬(P ∧ ¬Q) ↔Intro

1. $P \rightarrow Q$ for ↔Intro

2. $P \wedge \neg Q$ for ¬ Intro

3. P ∧Elim2

4. $\neg Q$ ∧Elim2

$\neg(P \wedge \neg Q)$

$\neg(P \wedge \neg Q)$ for ↔Intro

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$ ↔Intro

1. P→Q for ↔Intro

2. P∧¬Q for ¬ Intro

3. P ∧Elim2

4. ¬Q ∧Elim2

5. Q →Elim1,3

¬(P ∧ ¬Q)

¬(P ∧ ¬Q) for ↔Intro

P→Q

(P → Q) ↔ ¬(P ∧ ¬Q) ↔Intro

1. P→Q for ↔Intro

2. P∧¬Q for ¬ Intro

3. P ∧Elim2

4. ¬Q ∧Elim2

5. Q →Elim1,3

6. ⊥ ⊥Intro 4,5

¬(P ∧ ¬Q)

¬(P ∧ ¬Q) for ↔Intro

P→Q

(P → Q) ↔ ¬(P ∧ ¬Q) ↔Intro

1. $P \rightarrow Q$ for $\leftrightarrow$Intro

2. $P \wedge \neg Q$ for $\neg$ Intro

3. P $\wedge$Elim2

4. $\neg Q$ $\wedge$Elim2

5. Q $\rightarrow$Elim1,3

6. $\bot$ $\bot$Intro 4,5

7. $\neg(P \wedge \neg Q)$ $\neg$Intro 2-6

$\neg(P \wedge \neg Q)$ for $\leftrightarrow$Intro

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$ $\leftrightarrow$Intro

1. P→Q for ↔Intro

2. P∧¬Q for ¬ Intro

3. P ∧Elim2

4. ¬Q ∧Elim2

5. Q →Elim1,3

6. ⊥ ⊥Intro 4,5

7. ¬(P ∧ ¬Q) ¬Intro 2-6

8. ¬(P ∧ ¬Q) for ↔Intro

P→Q

(P → Q) ↔ ¬(P ∧ ¬Q) ↔Intro

1.	$P \to Q$	for $\leftrightarrow$Intro
2.	$P \wedge \neg Q$	for $\neg$ Intro
3.	P	$\wedge$Elim2
4.	$\neg Q$	$\wedge$Elim2
5.	Q	$\to$Elim1,3
6.	$\bot$	$\bot$Intro 4,5
7.	$\neg(P \wedge \neg Q)$	$\neg$Intro 2-6
8.	$\neg(P \wedge \neg Q)$	for $\leftrightarrow$Intro
9.	P	for $\to$Intro
	Q	
	$P \to Q$	$\to$Intro 9-
	$(P \to Q) \leftrightarrow \neg(P \wedge \neg Q)$	$\leftrightarrow$Intro

1.	$P \to Q$	for $\leftrightarrow$Intro
2.	$P \wedge \neg Q$	for $\neg$ Intro
3.	P	$\wedge$Elim2
4.	$\neg Q$	$\wedge$Elim2
5.	Q	$\to$Elim1,3
6.	$\bot$	$\bot$Intro 4,5
7.	$\neg(P \wedge \neg Q)$	$\neg$Intro 2-6
8.	$\neg(P \wedge \neg Q)$	for $\leftrightarrow$Intro
9.	P	for $\to$Intro
10.	$\neg Q$	for $\neg$ Intro
	Q	$\neg$ Intro
	$P \to Q$	$\to$Intro 9-
	$(P \to Q) \leftrightarrow \neg(P \wedge \neg Q)$	$\leftrightarrow$Intro

1. P→Q   for ↔Intro

2. P∧¬Q   for ¬ Intro

3. P   ∧Elim2

4. ¬Q   ∧Elim2

5. Q   →Elim1,3

6. ⊥   ⊥Intro 4,5

7. ¬(P ∧ ¬Q)   ¬Intro 2-6

8. ¬(P ∧ ¬Q)   for ↔Intro

9. P   for →Intro

10. ¬Q   for ¬ Intro

11. P∧¬Q   ∧Intro 9,10

Q   ¬ Intro

P→Q   →Intro 9-

(P → Q) ↔ ¬(P ∧ ¬Q)   ↔Intro

1. $P \rightarrow Q$	for $\leftrightarrow$Intro
2. $P \wedge \neg Q$	for $\neg$ Intro
3. P	$\wedge$Elim2
4. $\neg Q$	$\wedge$Elim2
5. Q	$\rightarrow$Elim1,3
6. $\bot$	$\bot$Intro 4,5
7. $\neg(P \wedge \neg Q)$	$\neg$Intro 2-6
8. $\neg(P \wedge \neg Q)$	for $\leftrightarrow$Intro
9. P	for $\rightarrow$Intro
10. $\neg Q$	for $\neg$ Intro
11. $P \wedge \neg Q$	$\wedge$Intro 9,10
12. $\bot$	$\bot$Intro 8,11
Q	$\neg$ Intro
$P \rightarrow Q$	$\rightarrow$Intro 9-
$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$	$\leftrightarrow$Intro

1.	$P \to Q$	for ↔Intro
2.	$P \wedge \neg Q$	for ¬ Intro
3.	P	∧Elim2
4.	$\neg Q$	∧Elim2
5.	Q	→Elim1,3
6.	$\bot$	⊥Intro 4,5
7.	$\neg(P \wedge \neg Q)$	¬Intro 2-6
8.	$\neg(P \wedge \neg Q)$	for ↔Intro
9.	P	for →Intro
10.	$\neg Q$	for ¬ Intro
11.	$P \wedge \neg Q$	∧Intro 9,10
12.	$\bot$	⊥Intro 8,11
13.	Q	¬ Intro 10-12
14.	$P \to Q$	→Intro 9-13
15.	$(P \to Q) \leftrightarrow \neg(P \wedge \neg Q)$	↔Intro 1-7, 8-14

Pushing Negations Inside

Pushing Negations Inside

DeMorgan's Laws

$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$

$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$

Pushing Negations Inside

DeMorgan's Laws

$\neg(P \lor Q) \Leftrightarrow (\neg P \land \neg Q)$

$\neg(P \land Q) \Leftrightarrow (\neg P \lor \neg Q)$

Negated Conditional

$\neg(P \to Q) \Leftrightarrow (P \land \neg Q)$

Pushing Negations Inside

DeMorgan's Laws

$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$

$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$

Negated Conditional

$\neg(P \rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$

Negated Biconditional

$\neg(P \leftrightarrow Q) \Leftrightarrow (\neg P \leftrightarrow Q)$

Pushing Negations Inside

DeMorgan's Laws

$\neg(P \lor Q) \Leftrightarrow (\neg P \land \neg Q)$

$\neg(P \land Q) \Leftrightarrow (\neg P \lor \neg Q)$

Negated Conditional

$\neg(P \rightarrow Q) \Leftrightarrow (P \land \neg Q)$

Negated Biconditional

$\neg(P \leftrightarrow Q) \Leftrightarrow (\neg P \leftrightarrow Q)$

With repeated applications of these rules, we can convert any sentence with main connective ¬ into something with a different main connective.

Pushing Negations Inside

DeMorgan's Laws

$\neg(P \lor Q) \Leftrightarrow (\neg P \land \neg Q)$

$\neg(P \land Q) \Leftrightarrow (\neg P \lor \neg Q)$

Negated Conditional

$\neg(P \rightarrow Q) \Leftrightarrow (P \land \neg Q)$

Negated Biconditional

$\neg(P \leftrightarrow Q) \Leftrightarrow (\neg P \leftrightarrow Q)$

With repeated applications of these rules, we can convert any sentence with main connective ¬ into something with a different main connective.

Or get rid of any particular connectives that we don't like

The Taut Con Rule

- If some sentence C really is a tautological consequence of some other sentences $P_1, P_2, \ldots$ the rule Taut Con allows you to infer C from $P_1, P_2, \ldots$

The Taut Con Rule

- If some sentence C really is a tautological consequence of some other sentences $P_1, P_2, \ldots$ the rule Taut Con allows you to infer C from $P_1, P_2, \ldots$

Example

1. $P \rightarrow Q$
2. $\neg Q$

---

3. $\neg P$ Taut Con 1,2

The Taut Con Rule

- If some sentence C really is a tautological consequence of some other sentences $P_1, P_2, \ldots$ the rule Taut Con allows you to infer C from $P_1, P_2, \ldots$

Example

1. $P \rightarrow Q$
2. $\neg Q$

---

3. $\neg P$ Taut Con 1,2

(Modus Tollens)

The Taut Con Rule

- If some sentence C really is a tautological consequence of some other sentences $P_1, P_2, \ldots$ the rule Taut Con allows you to infer C from $P_1, P_2, \ldots$

Example

1. $P \rightarrow Q$
2. $\neg Q$

---

3. $\neg P$ Taut Con 1,2

(Modus Tollens)

Example

1. $P \lor Q$
2. $\neg Q$

---

3. P Taut Con 1,2

The Taut Con Rule

- If some sentence C really is a tautological consequence of some other sentences P_1, P_2, ... the rule Taut Con allows you to infer C from P_1, P_2, ...

Example

1. $P \rightarrow Q$
2. $\neg Q$

---

3. $\neg P$ Taut Con 1,2

(Modus Tollens)

Example

1. $P \vee Q$
2. $\neg Q$

---

3. P Taut Con 1,2

(Disjunctive Syllogism)

The Taut Con Rule (in our class)

- On future homeworks (homeworks 6-9) I will allow you to use Taut Con for any step I consider to be sufficiently obvious. It will be useful to use for the following cases:

1. $P \rightarrow Q$
2. $\neg Q$

---

3. $\neg P$ Modus Tollens

1. $P \vee Q$
2. $\neg Q$

---

3. P DS

1. $P \leftrightarrow Q$
2. $\neg P$

---

3. $\neg Q$ Biconditional

The Taut Con Rule (in our class)

- Or to replace one sentence with something equivalent to it

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- Or to replace one sentence with something equivalent to it

DeMorgan's Laws

$\neg(P \lor Q) \Leftrightarrow (\neg P \land \neg Q)$

$\neg(P \land Q) \Leftrightarrow (\neg P \lor \neg Q)$

The Taut Con Rule (in our class)

- Or to replace one sentence with something equivalent to it

DeMorgan's Laws

$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$

$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$

Conditionals

$\neg(P \rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$

$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

The Taut Con Rule (in our class)

- Or to replace one sentence with something equivalent to it

DeMorgan's Laws

$\neg(P \lor Q) \Leftrightarrow (\neg P \land \neg Q)$

$\neg(P \land Q) \Leftrightarrow (\neg P \lor \neg Q)$

Biconditionals

$\neg(P \leftrightarrow Q) \Leftrightarrow (\neg P \leftrightarrow Q)$

$(P \leftrightarrow Q) \Leftrightarrow (\neg P \leftrightarrow \neg Q)$

Conditionals

$\neg(P \rightarrow Q) \Leftrightarrow (P \land \neg Q)$

$(P \rightarrow Q) \Leftrightarrow (\neg P \lor Q)$

Problem using Taut Con

$S \vee (P \leftrightarrow Q)$

$S \rightarrow R$

$P \vee (Q \rightarrow R)$

Problem using Taut Con

$S \lor (P \leftrightarrow Q)$

$S \to R$

$P \lor (Q \to R)$

1. $S \lor (P \leftrightarrow Q)$
2. $S \to R$

$P \lor (Q \to R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ for ¬I

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ for ¬I

4. $\neg P \wedge \neg(Q \rightarrow R)$ DeMorgans 3

$P \vee (Q \rightarrow R)$

1. $S \lor (P \leftrightarrow Q)$
2. $S \rightarrow R$

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3. $\neg(P \lor (Q \rightarrow R))$ for ¬I

---

4. $\neg P \land \neg(Q \rightarrow R)$ DeMorgans 3
5. $\neg P$ ∧Elim 4

$P \lor (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ for ¬I

4. $\neg P \wedge \neg(Q \rightarrow R)$ DeMorgans 3

5. $\neg P$ ∧Elim 4

6. $\neg(Q \rightarrow R)$ ∧Elim 4

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ for ¬I

4. $\neg P \wedge \neg(Q \rightarrow R)$ DeMorgans 3

5. $\neg P$ ∧Elim 4

6. $\neg(Q \rightarrow R)$ ∧Elim 4

7. $Q \wedge \neg R$ NegCon 6

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ for ¬I

4. $\neg P \wedge \neg(Q \rightarrow R)$ DeMorgans 3

5. $\neg P$ ∧Elim 4

6. $\neg(Q \rightarrow R)$ ∧Elim 4

7. $Q \wedge \neg R$ NegCon 6

8. Q ∧Elim 7

$P \vee (Q \rightarrow R)$

1. $S \lor (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \lor (Q \rightarrow R))$ for $\neg$I

4. $\neg P \land \neg(Q \rightarrow R)$ DeMorgans 3

5. $\neg P$ $\land$Elim 4

6. $\neg(Q \rightarrow R)$ $\land$Elim 4

7. $Q \land \neg R$ NegCon 6

8. Q $\land$Elim 7

9. $\neg R$ $\land$Elim 7

$P \lor (Q \rightarrow R)$

1. S ∨ (P↔Q)
2. S→R

3. ¬(P ∨ (Q→R)) for ¬I

4. ¬P ∧ ¬(Q→R) DeMorgans 3
5. ¬P ∧Elim 4
6. ¬(Q→R) ∧Elim 4
7. Q∧¬R NegCon 6
8. Q ∧Elim 7
9. ¬R ∧Elim 7
10. ¬S Modus Tollens 2,9

P ∨ (Q→R)

1. $S \vee (P \leftrightarrow Q)$
2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ for ¬I

4. $\neg P \wedge \neg(Q \rightarrow R)$ DeMorgans 3
5. $\neg P$ ∧Elim 4
6. $\neg(Q \rightarrow R)$ ∧Elim 4
7. $Q \wedge \neg R$ NegCon 6
8. Q ∧Elim 7
9. $\neg R$ ∧Elim 7
10. $\neg S$ Modus Tollens 2,9
11. $P \leftrightarrow Q$ Disjunctive Syllogism 1,10

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

---

3. $\neg(P \vee (Q \rightarrow R))$ for $\neg$I

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4. $\neg P \wedge \neg(Q \rightarrow R)$ DeMorgans 3

5. $\neg P$ $\wedge$Elim 4

6. $\neg(Q \rightarrow R)$ $\wedge$Elim 4

7. $Q \wedge \neg R$ NegCon 6

8. Q $\wedge$Elim 7

9. $\neg R$ $\wedge$Elim 7

10. $\neg S$ Modus Tollens 2,9

11. $P \leftrightarrow Q$ Disjunctive Syllogism 1,10

12. P $\leftrightarrow$Elim 8,11

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ — for ¬I

4. $\neg P \wedge \neg(Q \rightarrow R)$ — DeMorgans 3

5. $\neg P$ — ∧Elim 4

6. $\neg(Q \rightarrow R)$ — ∧Elim 4

7. $Q \wedge \neg R$ — NegCon 6

8. Q — ∧Elim 7

9. $\neg R$ — ∧Elim 7

10. $\neg S$ — Modus Tollens 2,9

11. $P \leftrightarrow Q$ — Disjunctive Syllogism 1,10

12. P — ↔Elim 8,11

13. $\bot$ — ⊥ Intro 5,12

$P \vee (Q \rightarrow R)$

Red text for Fitch

1. S ∨ (P↔Q)
2. S→R
3. ¬(P ∨ (Q→R)) for ¬I
4. ¬P ∧ ¬(Q→R) Taut Con 3 DeMorgans 3
5. ¬P ∧Elim 4
6. ¬(Q→R) ∧Elim 4
7. Q∧¬R Taut Con 6 NegCon 6
8. Q ∧Elim 7
9. ¬R ∧Elim 7
10. ¬S Taut Con 2,9 Modus Tollens 2,9
11. P↔Q Taut Con 1,10 Disjunctive Syllogism 1,10
12. P ↔Elim 11,12
13. ⊥ ⊥ Intro 5,12
14. P ∨ (Q→R) ¬Intro 3-13