

PUZZLE

On the island of knights and knaves, you meet A, B, and C.

A says "Either I am a knight or C is a knave"

B says "I am a knight if and only if C is a knave"

C says "B is a knave".

Who is what?

THE PARADOXES OF MATERIAL IMPLICATION

Monday, 3 March

PARADOXES OF MATERIAL IMPLICATION

Example:

$$\left| \begin{array}{l} \neg P \vee Q \\ \hline P \rightarrow Q \end{array} \right.$$

PARADOXES OF MATERIAL IMPLICATION

Example:

$$\begin{array}{|l} \neg P \vee Q \\ \hline P \rightarrow Q \end{array}$$

$$\text{I. } \neg P \vee Q$$

$$P \rightarrow Q$$

PARADOXES OF MATERIAL IMPLICATION

Example:

$$\frac{\neg P \vee Q}{P \rightarrow Q}$$
$$1. \neg P \vee Q$$
$$2. P$$

for \rightarrow Intro

$$Q$$
$$P \rightarrow Q \quad \rightarrow \text{Intro}$$

PARADOXES OF MATERIAL IMPLICATION

Example:

$$\frac{\neg P \vee Q}{P \rightarrow Q}$$

$$1. \neg P \vee Q$$

$$2. P$$

for \rightarrow Intro

Now disjunctive syllogism

$$Q$$

$$P \rightarrow Q \quad \rightarrow \text{Intro}$$

PARADOXES OF MATERIAL IMPLICATION

Example:

$$\frac{\neg P \vee Q}{P \rightarrow Q}$$

$$1. \neg P \vee Q$$

$$2. P$$

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$$Q$$

$$P \rightarrow Q \quad \rightarrow \text{Intro}$$

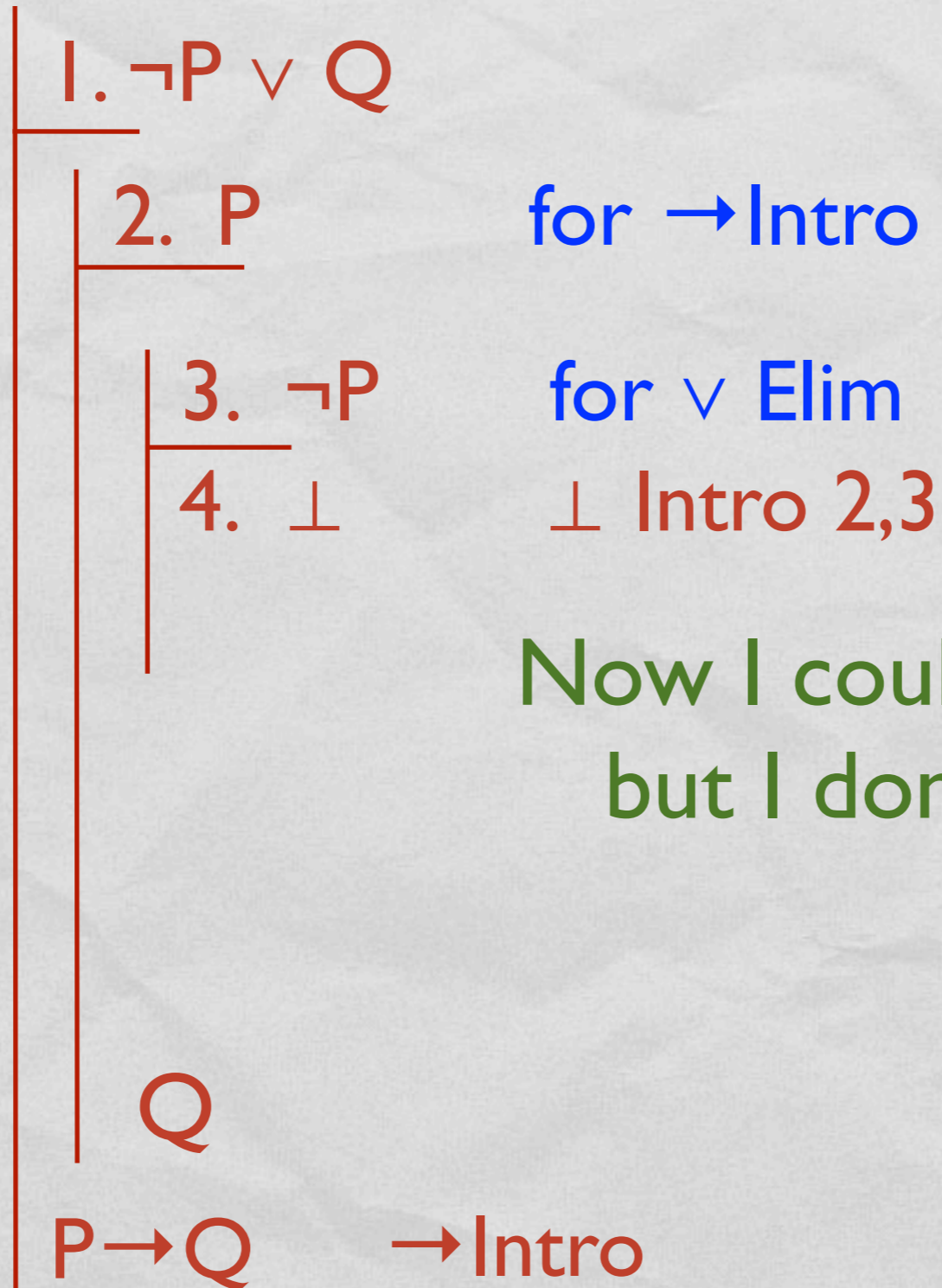
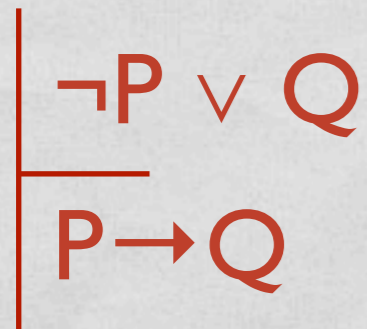
PARADOXES OF MATERIAL IMPLICATION

Example:

$$\frac{\neg P \vee Q}{P \rightarrow Q}$$
$$\begin{array}{l} \text{1. } \neg P \vee Q \\ \hline \text{2. } P \quad \text{for } \rightarrow \text{Intro} \\ \quad \text{3. } \neg P \quad \text{for } \vee \text{Elim} \\ \quad \quad \text{4. } \perp \quad \perp \text{Intro 2,3} \\ \quad \quad \quad Q \\ \hline P \rightarrow Q \quad \rightarrow \text{Intro} \end{array}$$

PARADOXES OF MATERIAL IMPLICATION

Example:



Now I could do \neg intro
but I don't want to

PARADOXES OF MATERIAL IMPLICATION

Example:

$$\frac{\neg P \vee Q}{P \rightarrow Q}$$
$$\begin{array}{l} \hline 1. \neg P \vee Q \\ \hline \begin{array}{l} 2. P \quad \text{for } \rightarrow \text{Intro} \\ \hline \begin{array}{l} 3. \neg P \quad \text{for } \vee \text{Elim} \\ \hline 4. \perp \quad \perp \text{Intro } 2,3 \end{array} \\ \hline Q \end{array} \\ \hline P \rightarrow Q \quad \rightarrow \text{Intro} \end{array}$$

The \perp elim rule says from \perp , infer anything

PARADOXES OF MATERIAL IMPLICATION

Example:

$$\frac{\neg P \vee Q}{P \rightarrow Q}$$

1. $\neg P \vee Q$	
2. P	for \rightarrow Intro
3. $\neg P$	for \vee Elim
4. \perp	\perp Intro 2,3
5. Q	\perp Elim 4
Q	
$P \rightarrow Q$	\rightarrow Intro

The \perp elim rule says from \perp , infer anything

PARADOXES OF MATERIAL IMPLICATION

Example:

$$\frac{\neg P \vee Q}{P \rightarrow Q}$$

1. $\neg P \vee Q$

2. P

for \rightarrow Intro

3. $\neg P$

for \vee Elim

4. \perp

\perp Intro 2,3

5. Q

\perp Elim 4

6. Q

for \vee Elim

Q

$P \rightarrow Q$ \rightarrow Intro

PARADOXES OF MATERIAL IMPLICATION

Example:

$$\frac{\neg P \vee Q}{P \rightarrow Q}$$

1. $\neg P \vee Q$

2. P

for \rightarrow Intro

3. $\neg P$

for \vee Elim

4. \perp

\perp Intro 2,3

5. Q

\perp Elim 4

6. Q

for \vee Elim

7. Q

\vee elim 1, 3-5, 6-6

8. $P \rightarrow Q$ \rightarrow Intro

PARADOXES OF MATERIAL IMPLICATION

Example:

$$\frac{\neg P \vee Q}{P \rightarrow Q}$$

1. $\neg P \vee Q$

2. P

for \rightarrow Intro

3. $\neg P$

for \vee Elim

4. \perp

\perp Intro 2,3

5. Q

\perp Elim 4

6. Q

for \vee Elim

7. Q

\vee elim 1, 3-5, 6-6

8. $P \rightarrow Q$ \rightarrow Intro 2-7

PARADOXES OF MATERIAL IMPLICATION

We just showed:

$$\frac{\neg P \vee Q}{P \rightarrow Q}$$

PARADOXES OF MATERIAL IMPLICATION

We just showed:

$$\frac{\neg P \vee Q}{P \rightarrow Q}$$

We did this by showing that:

$$\frac{\neg P}{P \rightarrow Q}$$

and

$$\frac{Q}{P \rightarrow Q}$$

PARADOXES OF MATERIAL IMPLICATION

We just showed:

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We did this by showing that:

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and

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$$\begin{array}{l} \frac{1. \neg P}{\begin{array}{l} \frac{2. P}{\begin{array}{l} 3. \perp \quad \perp \text{ Intro } 1,2 \\ 4. Q \quad \perp \text{ Elim } 3 \end{array}} \\ 5. P \rightarrow Q \quad \rightarrow \text{Intro } 2-4 \end{array}} \end{array}$$

PARADOXES OF MATERIAL IMPLICATION

We just showed:

$$\frac{\neg P \vee Q}{P \rightarrow Q}$$

We did this by showing that:

$$\frac{\neg P}{P \rightarrow Q}$$

and

$$\frac{Q}{P \rightarrow Q}$$

1. $\neg P$	
2. P	for \rightarrowIntro
3. \perp	\perp Intro 1,2
4. Q	\perp Elim 3
5. $P \rightarrow Q$	\rightarrow Intro 2-4

1. Q	
2. P	for \rightarrowIntro
3. Q	Reit 1
5. $P \rightarrow Q$	\rightarrow Intro 2-3

THE OTHER DIRECTION

Example:

$$\begin{array}{|l} P \rightarrow Q \\ \hline \neg P \vee Q \end{array}$$

THE OTHER DIRECTION

Example:

$$\begin{array}{|l} P \rightarrow Q \\ \hline \neg P \vee Q \end{array}$$

$$\text{I. } P \rightarrow Q$$

$$\neg P \vee Q$$

THE OTHER DIRECTION

Example:

$$\frac{P \rightarrow Q}{\neg P \vee Q}$$

$$1. P \rightarrow Q$$

$$2. \neg(\neg P \vee Q) \quad \text{for } \neg\text{-Intro}$$

\perp

\perp Intro

$$\neg P \vee Q$$

\neg -Intro 2-

THE OTHER DIRECTION

Example:

$$\frac{P \rightarrow Q}{\neg P \vee Q}$$

$$\begin{array}{l} \frac{1. P \rightarrow Q}{\frac{2. \neg(\neg P \vee Q) \quad \text{for } \neg\text{Intro}}{\frac{3. \neg P \quad \text{for } \neg\text{Intro}}{\perp} \quad \perp \text{Intro}}} \\ \neg P \vee Q \quad \neg\text{Intro 2-} \end{array}$$

THE OTHER DIRECTION

Example:

$$\frac{P \rightarrow Q}{\neg P \vee Q}$$

$$\begin{array}{l} \frac{1. P \rightarrow Q}{\quad} \\ \frac{2. \neg(\neg P \vee Q) \quad \text{for } \neg\text{Intro}}{\quad} \\ \frac{3. \neg P \quad \text{for } \neg\text{Intro}}{4. \neg P \vee Q \quad \vee\text{Intro } 3} \\ \perp \quad \perp \text{Intro} \\ \neg P \vee Q \quad \neg\text{Intro } 2- \end{array}$$

THE OTHER DIRECTION

Example:

$$\frac{P \rightarrow Q}{\neg P \vee Q}$$

1. $P \rightarrow Q$

2. $\neg(\neg P \vee Q)$ for \neg Intro

3. $\neg P$ for \neg Intro

4. $\neg P \vee Q$ \vee Intro 3

5. \perp \perp Intro 2,4

\perp \perp Intro

$\neg P \vee Q$ \neg Intro 2-

THE OTHER DIRECTION

Example:

$$\frac{P \rightarrow Q}{\neg P \vee Q}$$

$$1. P \rightarrow Q$$

$$2. \neg(\neg P \vee Q) \quad \text{for } \neg\text{Intro}$$

$$3. \neg P \quad \text{for } \neg\text{Intro}$$

$$4. \neg P \vee Q \quad \vee\text{Intro } 3$$

$$5. \perp \quad \perp \text{Intro } 2,4$$

$$6. P \quad \neg\text{Intro } 3-5$$

$$\perp \quad \perp \text{Intro}$$

$$\neg P \vee Q \quad \neg\text{Intro } 2-$$

THE OTHER DIRECTION

Example:

$$\frac{P \rightarrow Q}{\neg P \vee Q}$$

1. $P \rightarrow Q$

2. $\neg(\neg P \vee Q)$ for \neg Intro

3. $\neg P$ for \neg Intro

4. $\neg P \vee Q$ \vee Intro 3

5. \perp \perp Intro 2,4

6. P \neg Intro 3-5

7. Q \rightarrow Elim 1,6

\perp \perp Intro

$\neg P \vee Q$ \neg Intro 2-

THE OTHER DIRECTION

Example:

$$\frac{P \rightarrow Q}{\neg P \vee Q}$$

$$1. P \rightarrow Q$$

$$2. \neg(\neg P \vee Q) \quad \text{for } \neg\text{Intro}$$

$$3. \neg P \quad \text{for } \neg\text{Intro}$$

$$4. \neg P \vee Q \quad \vee\text{Intro } 3$$

$$5. \perp \quad \perp \text{Intro } 2,4$$

$$6. P \quad \neg\text{Intro } 3-5$$

$$7. Q \quad \rightarrow\text{Elim } 1,6$$

$$8. \neg P \vee Q \quad \vee\text{Intro } 7$$

$$\perp \quad \perp \text{Intro}$$

$$\neg P \vee Q \quad \neg\text{Intro } 2-$$

THE OTHER DIRECTION

Example:

$$\frac{P \rightarrow Q}{\neg P \vee Q}$$

$$1. P \rightarrow Q$$

$$2. \neg(\neg P \vee Q) \quad \text{for } \neg\text{Intro}$$

$$3. \neg P \quad \text{for } \neg\text{Intro}$$

$$4. \neg P \vee Q \quad \vee\text{Intro } 3$$

$$5. \perp \quad \perp \text{Intro } 2,4$$

$$6. P \quad \neg\text{Intro } 3-5$$

$$7. Q \quad \rightarrow\text{Elim } 1,6$$

$$8. \neg P \vee Q \quad \vee\text{Intro } 7$$

$$9. \perp \quad \perp \text{Intro } 2,8$$

$$10. \neg P \vee Q \quad \neg\text{Intro } 2-9$$

PROVING BICONDITIONALS

We have now proved:

$$\left| \begin{array}{l} P \rightarrow Q \\ \hline \neg P \vee Q \end{array} \right.$$

PROVING BICONDITIONALS

We have now proved:

$$\left| \begin{array}{l} P \rightarrow Q \\ \hline \neg P \vee Q \end{array} \right. \quad \text{and} \quad \left| \begin{array}{l} \neg P \vee Q \\ \hline P \rightarrow Q \end{array} \right.$$

PROVING BICONDITIONALS

We have now proved:

$$\left| \begin{array}{l} P \rightarrow Q \\ \hline \neg P \vee Q \end{array} \right. \quad \text{and} \quad \left| \begin{array}{l} \neg P \vee Q \\ \hline P \rightarrow Q \end{array} \right.$$

Therefore we could prove:

$$\left| \begin{array}{l} \hline (P \rightarrow Q) \leftrightarrow (\neg P \vee Q) \end{array} \right.$$

PROVING BICONDITIONALS

$$(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$$

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$$(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$$

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PROVING BICONDITIONALS

$$\frac{}{(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)}$$

I. $P \rightarrow Q$ for \leftrightarrow Intro

$\neg P \vee Q$

$\neg P \vee Q$ for \leftrightarrow Intro

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$ \leftrightarrow Intro

I. $P \rightarrow Q$

$\neg P \vee Q$

$\neg P \vee Q$

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$

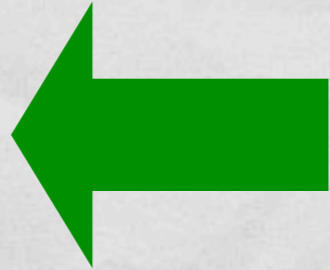
1. $P \rightarrow Q$

$\neg P \vee Q$

$\neg P \vee Q$

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$



1. $P \rightarrow Q$

2. $\neg(\neg P \vee Q)$ for \neg Intro

3. $\neg P$ for \neg Intro

4. $\neg P \vee Q$ \vee Intro 3

5. \perp \perp Intro 2,4

6. P \neg Intro 3-5

7. Q \rightarrow Elim 1,6

8. $\neg P \vee Q$ \vee Intro 7

9. \perp \perp Intro 2,7

10. $\neg P \vee Q$ \neg Intro 2-9

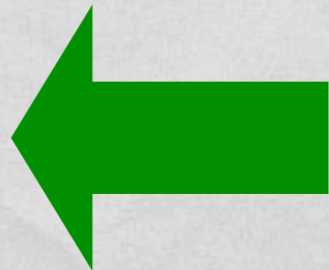
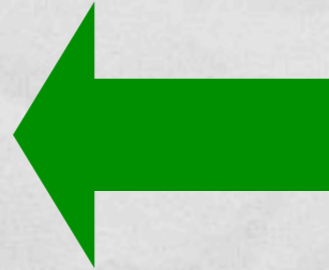
1. $P \rightarrow Q$

$\neg P \vee Q$

$\neg P \vee Q$

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$



1. $P \rightarrow Q$

2. $\neg(\neg P \vee Q)$ for \neg Intro

3. $\neg P$ for \neg Intro

4. $\neg P \vee Q$ \vee Intro 3

5. \perp \perp Intro 2,4

6. P \neg Intro 3-5

7. Q \rightarrow Elim 1,6

8. $\neg P \vee Q$ \vee Intro 7

9. \perp \perp Intro 2,7

10. $\neg P \vee Q$ \neg Intro 2-9

1. $\neg P \vee Q$

2. P for \rightarrow Intro

3. $\neg P$ for \vee Elim

4. \perp \perp Intro 2,3

5. Q \perp Elim 4

6. Q for \vee Elim

7. Q \vee Elim 1,3-5,6-6

8. $P \rightarrow Q$ \rightarrow Intro 2-7

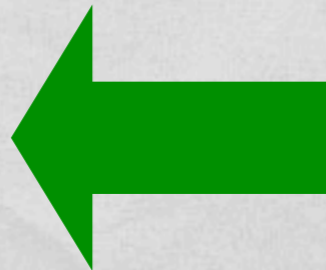
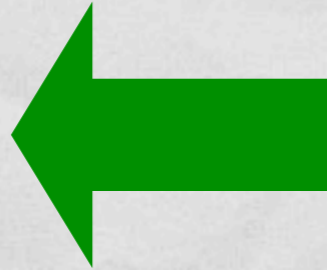
1. $P \rightarrow Q$

$\neg P \vee Q$

$\neg P \vee Q$

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$



1. $P \rightarrow Q$

2. $\neg(\neg P \vee Q)$ for \neg Intro

3. $\neg P$ for \neg Intro

4. $\neg P \vee Q$ \vee Intro 3

5. \perp \perp Intro 2,4

6. P \neg Intro 3-5

7. Q \rightarrow Elim 1,6

8. $\neg P \vee Q$ \vee Intro 7

9. \perp \perp Intro 2,7

10. $\neg P \vee Q$ \neg Intro 2-9

1. $\neg P \vee Q$

2. P for \rightarrow Intro

3. $\neg P$ for \vee Elim

4. \perp \perp Intro 2,3

5. Q \perp Elim 4

6. Q for \vee Elim

7. Q \vee Elim 1,3-5,6-6

8. $P \rightarrow Q$ \rightarrow Intro 2-7

\leftrightarrow Intro 1-10, 11-18

BICONDITIONALS AND EQUIVALENCE

We have now proved:

$$\left| \begin{array}{l} \hline (P \rightarrow Q) \leftrightarrow (\neg P \vee Q) \end{array} \right.$$

If a biconditional is a logical truth then the two parts are logically equivalent:

$$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$

CHAINS OF EQUIVALENCE

$$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$

CHAINS OF EQUIVALENCE

$$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$

By DeMorgan's

$$(\neg P \vee Q) \Leftrightarrow \neg(P \wedge \neg Q)$$

CHAINS OF EQUIVALENCE

$$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$

By DeMorgan's

$$(\neg P \vee Q) \Leftrightarrow \neg(P \wedge \neg Q)$$

Therefore

$$(P \rightarrow Q) \Leftrightarrow \neg(P \wedge \neg Q)$$

CHAINS OF EQUIVALENCE

$$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$

By DeMorgan's

$$(\neg P \vee Q) \Leftrightarrow \neg(P \wedge \neg Q)$$

Therefore

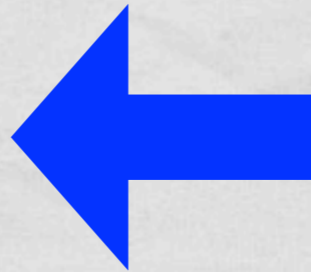
$$(P \rightarrow Q) \Leftrightarrow \neg(P \wedge \neg Q)$$

and so

$$\neg(P \rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$$

CHAINS OF EQUIVALENCE

$$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$



When a conditional is true

By DeMorgan's

$$(\neg P \vee Q) \Leftrightarrow \neg(P \wedge \neg Q)$$

Therefore

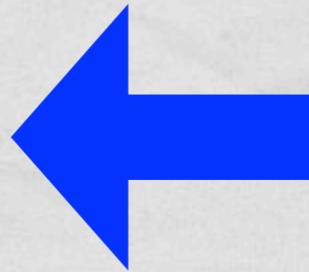
$$(P \rightarrow Q) \Leftrightarrow \neg(P \wedge \neg Q)$$

and so

$$\neg(P \rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$$

CHAINS OF EQUIVALENCE

$$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$



When a conditional is true

By DeMorgan's

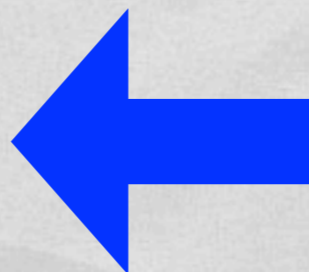
$$(\neg P \vee Q) \Leftrightarrow \neg(P \wedge \neg Q)$$

Therefore

$$(P \rightarrow Q) \Leftrightarrow \neg(P \wedge \neg Q)$$

and so

$$\neg(P \rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$$



When a conditional is false

NEGATED CONDITIONALS

$$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$$

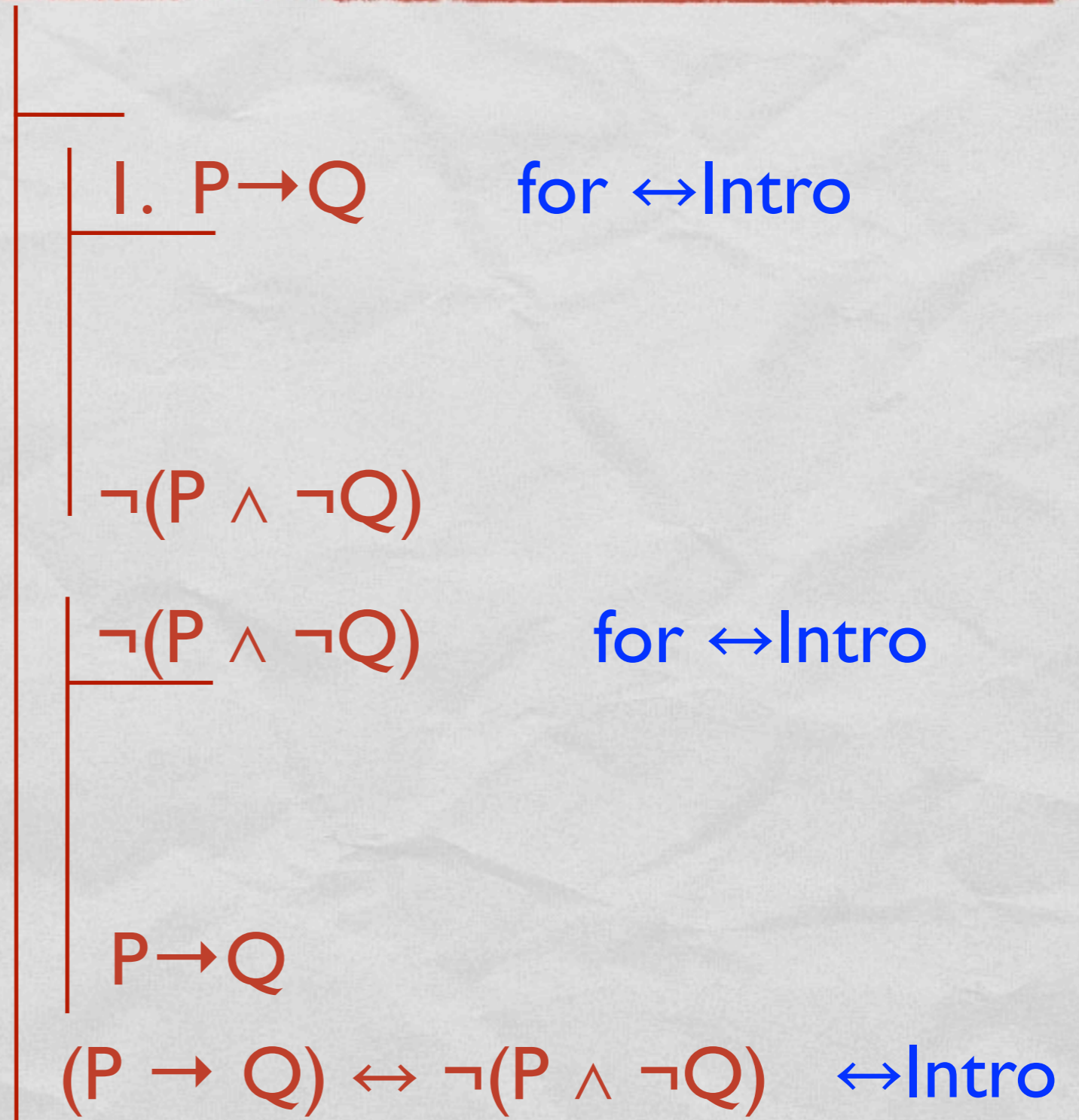
NEGATED CONDITIONALS

$$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$$

$$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$$

NEGATED CONDITIONALS

$$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$$



I. $P \rightarrow Q$

for \leftrightarrow Intro

$\neg(P \wedge \neg Q)$

$\neg(P \wedge \neg Q)$

for \leftrightarrow Intro

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$ \leftrightarrow Intro

1. $P \rightarrow Q$ for \leftrightarrow Intro

2. $P \wedge \neg Q$ for \neg Intro

$\neg(P \wedge \neg Q)$

$\neg(P \wedge \neg Q)$ for \leftrightarrow Intro

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$ \leftrightarrow Intro

1. $P \rightarrow Q$ for \leftrightarrow Intro

2. $P \wedge \neg Q$ for \neg Intro

3. P \wedge Elim2

$\neg(P \wedge \neg Q)$

$\neg(P \wedge \neg Q)$ for \leftrightarrow Intro

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$ \leftrightarrow Intro

1. $P \rightarrow Q$ for \leftrightarrow Intro

2. $P \wedge \neg Q$ for \neg Intro

3. P \wedge Elim2

4. $\neg Q$ \wedge Elim2

$\neg(P \wedge \neg Q)$

$\neg(P \wedge \neg Q)$ for \leftrightarrow Intro

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$ \leftrightarrow Intro

1. $P \rightarrow Q$ for \leftrightarrow Intro

2. $P \wedge \neg Q$ for \neg Intro

3. P \wedge Elim2

4. $\neg Q$ \wedge Elim2

5. Q \rightarrow Elim 1,3

$\neg(P \wedge \neg Q)$

$\neg(P \wedge \neg Q)$ for \leftrightarrow Intro

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$ \leftrightarrow Intro

1. $P \rightarrow Q$ for \leftrightarrow Intro

2. $P \wedge \neg Q$ for \neg Intro

3. P \wedge Elim2

4. $\neg Q$ \wedge Elim2

5. Q \rightarrow Elim 1,3

6. \perp \perp Intro 4,5

$\neg(P \wedge \neg Q)$

$\neg(P \wedge \neg Q)$ for \leftrightarrow Intro

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$ \leftrightarrow Intro

1. $P \rightarrow Q$ for \leftrightarrow Intro

2. $P \wedge \neg Q$ for \neg Intro

3. P \wedge Elim2

4. $\neg Q$ \wedge Elim2

5. Q \rightarrow Elim 1,3

6. \perp \perp Intro 4,5

7. $\neg(P \wedge \neg Q)$ \neg Intro 2-6

$\neg(P \wedge \neg Q)$ for \leftrightarrow Intro

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$ \leftrightarrow Intro

1. $P \rightarrow Q$ for \leftrightarrow Intro

2. $P \wedge \neg Q$ for \neg Intro

3. P \wedge Elim2

4. $\neg Q$ \wedge Elim2

5. Q \rightarrow Elim 1,3

6. \perp \perp Intro 4,5

7. $\neg(P \wedge \neg Q)$ \neg Intro 2-6

8. $\neg(P \wedge \neg Q)$ for \leftrightarrow Intro

$P \rightarrow Q$

$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$ \leftrightarrow Intro

1. $P \rightarrow Q$ for \leftrightarrow Intro

2. $P \wedge \neg Q$ for \neg Intro

3. P \wedge Elim2

4. $\neg Q$ \wedge Elim2

5. Q \rightarrow Elim 1,3

6. \perp \perp Intro 4,5

7. $\neg(P \wedge \neg Q)$ \neg Intro 2-6

8. $\neg(P \wedge \neg Q)$ for \leftrightarrow Intro

9. P for \rightarrow Intro

Q

$P \rightarrow Q$

\rightarrow Intro 9-

$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$ \leftrightarrow Intro

1. $P \rightarrow Q$	for \leftrightarrow Intro
2. $P \wedge \neg Q$	for \neg Intro
3. P	\wedge Elim2
4. $\neg Q$	\wedge Elim2
5. Q	\rightarrow Elim 1,3
6. \perp	\perp Intro 4,5
7. $\neg(P \wedge \neg Q)$	\neg Intro 2-6
8. $\neg(P \wedge \neg Q)$	for \leftrightarrow Intro
9. P	for \rightarrow Intro
10. $\neg Q$	for \neg Intro
Q	\neg Intro
$P \rightarrow Q$	\rightarrow Intro 9-
$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$	\leftrightarrow Intro

1. $P \rightarrow Q$	for \leftrightarrow Intro
2. $P \wedge \neg Q$	for \neg Intro
3. P	\wedge Elim2
4. $\neg Q$	\wedge Elim2
5. Q	\rightarrow Elim 1,3
6. \perp	\perp Intro 4,5
7. $\neg(P \wedge \neg Q)$	\neg Intro 2-6
8. $\neg(P \wedge \neg Q)$	for \leftrightarrow Intro
9. P	for \rightarrow Intro
10. $\neg Q$	for \neg Intro
11. $P \wedge \neg Q$	\wedge Intro 9,10
Q	\neg Intro
$P \rightarrow Q$	\rightarrow Intro 9-
$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$	\leftrightarrow Intro

1. $P \rightarrow Q$	for \leftrightarrow Intro
2. $P \wedge \neg Q$	for \neg Intro
3. P	\wedge Elim2
4. $\neg Q$	\wedge Elim2
5. Q	\rightarrow Elim 1,3
6. \perp	\perp Intro 4,5
7. $\neg(P \wedge \neg Q)$	\neg Intro 2-6
8. $\neg(P \wedge \neg Q)$	for \leftrightarrow Intro
9. P	for \rightarrow Intro
10. $\neg Q$	for \neg Intro
11. $P \wedge \neg Q$	\wedge Intro 9,10
12. \perp	\perp Intro 8,11
Q	\neg Intro
$P \rightarrow Q$	\rightarrow Intro 9-
$(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$	\leftrightarrow Intro

1. $P \rightarrow Q$ for \leftrightarrow Intro

2. $P \wedge \neg Q$ for \neg Intro

3. P \wedge Elim2

4. $\neg Q$ \wedge Elim2

5. Q \rightarrow Elim 1,3

6. \perp \perp Intro 4,5

7. $\neg(P \wedge \neg Q)$ \neg Intro 2-6

8. $\neg(P \wedge \neg Q)$ for \leftrightarrow Intro

9. P for \rightarrow Intro

10. $\neg Q$ for \neg Intro

11. $P \wedge \neg Q$ \wedge Intro 9,10

12. \perp \perp Intro 8,11

13. Q \neg Intro 10-12

14. $P \rightarrow Q$ \rightarrow Intro 9-13

15. $(P \rightarrow Q) \leftrightarrow \neg(P \wedge \neg Q)$ \leftrightarrow Intro 1-7, 8-14

PUSHING NEGATIONS INSIDE

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DeMorgan's Laws

$$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$$

$$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$$

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$$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$$

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$$\neg(P \rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$$

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With repeated applications of these rules, we can convert any sentence with main connective \neg into something with a different main connective.

PUSHING NEGATIONS INSIDE

DeMorgan's Laws

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Negated Biconditional

$$\neg(P \leftrightarrow Q) \Leftrightarrow (\neg P \leftrightarrow Q)$$

With repeated applications of these rules, we can convert any sentence with main connective \neg into something with a different main connective.

Or get rid of any particular connectives that we don't like

THE TAUT CON RULE

- If some sentence C really is a tautological consequence of some other sentences P_1, P_2, \dots the rule Taut Con allows you to infer C from P_1, P_2, \dots

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Example

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2. $\neg Q$

3. $\neg P$

Taut Con 1,2

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(Modus Tollens)

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(Modus Tollens)

Example

1. $P \vee Q$	
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3. $\neg P$	Taut Con 1,2

(Modus Tollens)

Example

1. $P \vee Q$	
2. $\neg Q$	
<hr/>	
3. P	Taut Con 1,2

(Disjunctive Syllogism)

THE TAUT CON RULE (IN OUR CLASS)

- On future homeworks (homeworks 6-9) I will allow you to use Taut Con for any step I consider to be sufficiently obvious. It will be useful to use for the following cases:

$$\begin{array}{l} 1. P \rightarrow Q \\ 2. \neg Q \\ \hline 3. \neg P \end{array} \quad \text{Modus Tollens}$$

$$\begin{array}{l} 1. P \vee Q \\ 2. \neg Q \\ \hline 3. P \end{array} \quad \text{DS}$$

$$\begin{array}{l} 1. P \leftrightarrow Q \\ 2. \neg P \\ \hline 3. \neg Q \end{array} \quad \text{Biconditional}$$

THE TAUT CON RULE (IN OUR CLASS)

- Or to replace one sentence with something equivalent to it

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$$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$$

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THE TAUT CON RULE (IN OUR CLASS)

- Or to replace one sentence with something equivalent to it

DeMorgan's Laws

$$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$$

$$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$$

Conditionals

$$\neg(P \rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$$

$$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$

THE TAUT CON RULE (IN OUR CLASS)

- Or to replace one sentence with something equivalent to it

DeMorgan's Laws

$$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$$

$$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$$

Biconditionals

$$\neg(P \leftrightarrow Q) \Leftrightarrow (\neg P \leftrightarrow Q)$$

$$(P \leftrightarrow Q) \Leftrightarrow (\neg P \leftrightarrow \neg Q)$$

Conditionals

$$\neg(P \rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$$

$$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$

PROBLEM USING TAUT CON

$$\begin{array}{|l} S \vee (P \leftrightarrow Q) \\ S \rightarrow R \\ \hline P \vee (Q \rightarrow R) \end{array}$$

PROBLEM USING TAUT CON

$$\begin{array}{|l} S \vee (P \leftrightarrow Q) \\ S \rightarrow R \\ \hline P \vee (Q \rightarrow R) \end{array}$$

$$\begin{array}{|l} 1. S \vee (P \leftrightarrow Q) \\ 2. S \rightarrow R \\ \hline \end{array}$$

$$P \vee (Q \rightarrow R)$$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ for $\neg I$

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ for $\neg I$

4. $\neg P \wedge \neg(Q \rightarrow R)$ DeMorgans 3

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ for $\neg I$

4. $\neg P \wedge \neg(Q \rightarrow R)$ DeMorgans 3

5. $\neg P$ \wedge Elim 4

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ for $\neg I$

4. $\neg P \wedge \neg(Q \rightarrow R)$ DeMorgans 3

5. $\neg P$ \wedge Elim 4

6. $\neg(Q \rightarrow R)$ \wedge Elim 4

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ for $\neg I$

4. $\neg P \wedge \neg(Q \rightarrow R)$ DeMorgans 3

5. $\neg P$ \wedge Elim 4

6. $\neg(Q \rightarrow R)$ \wedge Elim 4

7. $Q \wedge \neg R$ NegCon 6

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ for $\neg I$

4. $\neg P \wedge \neg(Q \rightarrow R)$ DeMorgans 3

5. $\neg P$ \wedge Elim 4

6. $\neg(Q \rightarrow R)$ \wedge Elim 4

7. $Q \wedge \neg R$ NegCon 6

8. Q \wedge Elim 7

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ for $\neg I$

4. $\neg P \wedge \neg(Q \rightarrow R)$ DeMorgans 3

5. $\neg P$ \wedge Elim 4

6. $\neg(Q \rightarrow R)$ \wedge Elim 4

7. $Q \wedge \neg R$ NegCon 6

8. Q \wedge Elim 7

9. $\neg R$ \wedge Elim 7

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ for $\neg I$

4. $\neg P \wedge \neg(Q \rightarrow R)$ DeMorgans 3

5. $\neg P$ \wedge Elim 4

6. $\neg(Q \rightarrow R)$ \wedge Elim 4

7. $Q \wedge \neg R$ NegCon 6

8. Q \wedge Elim 7

9. $\neg R$ \wedge Elim 7

10. $\neg S$ Modus Tollens 2,9

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ for $\neg I$

4. $\neg P \wedge \neg(Q \rightarrow R)$ DeMorgans 3

5. $\neg P$ \wedge Elim 4

6. $\neg(Q \rightarrow R)$ \wedge Elim 4

7. $Q \wedge \neg R$ NegCon 6

8. Q \wedge Elim 7

9. $\neg R$ \wedge Elim 7

10. $\neg S$ Modus Tollens 2,9

11. $P \leftrightarrow Q$ Disjunctive Syllogism 1,10

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ for $\neg I$

4. $\neg P \wedge \neg(Q \rightarrow R)$ DeMorgans 3

5. $\neg P$ \wedge Elim 4

6. $\neg(Q \rightarrow R)$ \wedge Elim 4

7. $Q \wedge \neg R$ NegCon 6

8. Q \wedge Elim 7

9. $\neg R$ \wedge Elim 7

10. $\neg S$ Modus Tollens 2,9

11. $P \leftrightarrow Q$ Disjunctive Syllogism 1,10

12. P \leftrightarrow Elim 8,11

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

3. $\neg(P \vee (Q \rightarrow R))$ for $\neg I$

4. $\neg P \wedge \neg(Q \rightarrow R)$ DeMorgans 3

5. $\neg P$ \wedge Elim 4

6. $\neg(Q \rightarrow R)$ \wedge Elim 4

7. $Q \wedge \neg R$ NegCon 6

8. Q \wedge Elim 7

9. $\neg R$ \wedge Elim 7

10. $\neg S$ Modus Tollens 2,9

11. $P \leftrightarrow Q$ Disjunctive Syllogism 1,10

12. P \leftrightarrow Elim 8,11

13. \perp \perp Intro 5,12

$P \vee (Q \rightarrow R)$

1. $S \vee (P \leftrightarrow Q)$

2. $S \rightarrow R$

Red text for Fitch

3. $\neg(P \vee (Q \rightarrow R))$ for $\neg I$

4. $\neg P \wedge \neg(Q \rightarrow R)$ Taut Con 3 DeMorgans 3

5. $\neg P$ \wedge Elim 4

6. $\neg(Q \rightarrow R)$ \wedge Elim 4

7. $Q \wedge \neg R$ Taut Con 6 NegCon 6

8. Q \wedge Elim 7

9. $\neg R$ \wedge Elim 7

10. $\neg S$ Taut Con 2,9 Modus Tollens 2,9

11. $P \leftrightarrow Q$ Taut Con 1,10 Disjunctive Syllogism 1,10

12. P \leftrightarrow Elim 11,12

13. \perp \perp Intro 5,12

14. $P \vee (Q \rightarrow R)$ \neg Intro 3-13