Which are provable?

$P \rightarrow R$

$\vdash (P \wedge Q) \rightarrow R$

$P \rightarrow R$

$\vdash (P \vee Q) \rightarrow R$

$(P \wedge Q) \rightarrow R$

$\vdash P \rightarrow R$

$(P \vee Q) \rightarrow R$

$\vdash P \rightarrow R$

WHICH ARE PROVABLE?

$P \rightarrow R$

$(P \wedge Q) \rightarrow R$ VALID

$P \rightarrow R$

$(P \vee Q) \rightarrow R$ INVALID

$(P \wedge Q) \rightarrow R$

$P \rightarrow R$ INVALID

$(P \vee Q) \rightarrow R$

$P \rightarrow R$ VALID

PROOFS WITH CONDITIONALS III

Friday, 28 February

Rules for Conditionals

- → Elimination: from P → Q and P, we can infer Q.

1.	P → Q	
2.	P	
3.	Q	→ Elim: 1,2

- ↔ Elimination: from P ↔ Q and P/Q, we can infer Q/P.

1.	P ↔ Q	
2.	Q	
3.	P	↔ Elim: 1,2

Formal Proof Rules

- <u>→ Introduction</u>

  From a proof from P to Q, we can infer P → Q.

```
| | 1. P
| |------
| | ...
| | j. Q
| k. P → Q        → Intro: 1-j
```

This rule is often known as Conditional Proof

Think Main Connective

Example:

$P \rightarrow T$

$S \leftrightarrow T$

$(S \leftrightarrow R) \rightarrow (P \rightarrow R))$

Think Main Connective

Example:

| $P \rightarrow T$
| $S \leftrightarrow T$
| ---
| $(S \leftrightarrow R) \rightarrow (P \rightarrow R))$

1. $P \rightarrow T$
2. $S \leftrightarrow T$

$(S \leftrightarrow R) \rightarrow (P \rightarrow R))$

Think Main Connective

Example:

$P \rightarrow T$

$S \leftrightarrow T$

$(S \leftrightarrow R) \rightarrow (P \rightarrow R))$

1. $P \rightarrow T$
2. $S \leftrightarrow T$

---

3. $S \leftrightarrow R$ for $\rightarrow$Intro

$P \rightarrow R$

$(S \leftrightarrow R) \rightarrow (P \rightarrow R))$ by $\rightarrow$Intro 3-

Think Main Connective

Example:

$P \rightarrow T$

$S \leftrightarrow T$

$(S \leftrightarrow R) \rightarrow (P \rightarrow R))$

1. $P \rightarrow T$
2. $S \leftrightarrow T$

---

3. $S \leftrightarrow R$ for $\rightarrow$Intro

---

4. P for $\rightarrow$Intro

R

$P \rightarrow R$ by $\rightarrow$Intro 4-

$(S \leftrightarrow R) \rightarrow (P \rightarrow R))$ by $\rightarrow$Intro 3-

THINK MAIN CONNECTIVE

Example:

$P \rightarrow T$

$S \leftrightarrow T$

$(S \leftrightarrow R) \rightarrow (P \rightarrow R))$

1. $P \rightarrow T$
2. $S \leftrightarrow T$

---

3. $S \leftrightarrow R$ — for $\rightarrow$Intro

---

4. P — for $\rightarrow$Intro

---

5. T — $\rightarrow$Elim 1,5

R

$P \rightarrow R$ — by $\rightarrow$Intro 4-

$(S \leftrightarrow R) \rightarrow (P \rightarrow R))$ — by $\rightarrow$Intro 3-

Think Main Connective

Example:

P → T

S↔T

(S↔R)→(P→R))

1. P → T
2. S↔T

---

3. S↔R — for →Intro

---

4. P — for →Intro

---

5. T — →Elim 1,5

6. S — ↔Elim 2,5

R

P→R — by →Intro 4-

(S↔R)→(P→R)) — by →Intro 3-

Think Main Connective

Example:

P →T	
S↔T	
(S↔R)→(P→R))	

1. P →T	
2. S↔T	
3. S↔R	for →Intro
4. P	for →Intro
5. T	→Elim 1,5
6. S	↔Elim 2,5
7. R	↔Elim 3,6
P→R	by →Intro 4-
(S↔R)→(P→R))	by →Intro 3-

THINK MAIN CONNECTIVE

Example:

P → T

S↔T

(S↔R)→(P→R))

1. P → T
2. S↔T

3. S↔R for →Intro

4. P for →Intro

5. T →Elim 1,4

6. S ↔Elim 2,5

7. R ↔Elim 3,6

8. P→R →Intro 4-7

(S↔R)→(P→R)) by →Intro 3-

Think Main Connective

Example:

| P → T
| S↔T
|———
| (S↔R)→(P→R))

1. P → T
2. S↔T

———

3. S↔R — for →Intro

4. P — for →Intro

5. T — →Elim 1,4
6. S — ↔Elim 2,5
7. R — ↔Elim 3,6

8. P→R — →Intro 4-7

9. (S↔R)→(P→R)) — →Intro 3-8

Think Backwards

Example:

| $(P \to Q) \to R$

$(P \leftrightarrow Q) \to R$

1. $(P \to Q) \to R$

$(P \leftrightarrow Q) \to R$

Think Backwards

Example:

$(P \rightarrow Q) \rightarrow R$

$(P \leftrightarrow Q) \rightarrow R$

1. $(P \rightarrow Q) \rightarrow R$

2. $P \leftrightarrow Q$ for $\rightarrow$Intro

R

$(P \leftrightarrow Q) \rightarrow R$ $\rightarrow$Intro 2-

Think Backwards

Example:

$(P \rightarrow Q) \rightarrow R$

$(P \leftrightarrow Q) \rightarrow R$

1. $(P \rightarrow Q) \rightarrow R$

2. $P \leftrightarrow Q$ for $\rightarrow$Intro

How to get R?

R

$(P \leftrightarrow Q) \rightarrow R$ $\rightarrow$Intro 2-

Think Backwards

Example:

$(P \rightarrow Q) \rightarrow R$

$(P \leftrightarrow Q) \rightarrow R$

1. $(P \rightarrow Q) \rightarrow R$

2. $P \leftrightarrow Q$ for $\rightarrow$Intro

How to get R?
From line 1

R

$(P \leftrightarrow Q) \rightarrow R$ $\rightarrow$Intro 2-

THINK BACKWARDS

Example:

$(P \rightarrow Q) \rightarrow R$

$(P \leftrightarrow Q) \rightarrow R$

1. $(P \rightarrow Q) \rightarrow R$

2. $P \leftrightarrow Q$ for $\rightarrow$Intro

How to get R?
From line 1

$P \rightarrow Q$

R $\rightarrow$Elim 1,

$(P \leftrightarrow Q) \rightarrow R$ $\rightarrow$Intro 2-

Think Backwards

Example:

$(P \rightarrow Q) \rightarrow R$

$(P \leftrightarrow Q) \rightarrow R$

1. $(P \rightarrow Q) \rightarrow R$

---

2. $P \leftrightarrow Q$ — for $\rightarrow$ Intro

$P \rightarrow Q$

R — $\rightarrow$ Elim 1,

$(P \leftrightarrow Q) \rightarrow R$ — $\rightarrow$ Intro 2-

Think Backwards

Example:

$(P \to Q) \to R$

$(P \leftrightarrow Q) \to R$

1. $(P \to Q) \to R$

2. $P \leftrightarrow Q$ for $\to$ Intro

3. P for $\to$ Intro

Q

$P \to Q$ $\to$ Intro 3-

R $\to$ Elim 1,

$(P \leftrightarrow Q) \to R$ $\to$ Intro 2-

THINK BACKWARDS

Example:

$(P \rightarrow Q) \rightarrow R$

$(P \leftrightarrow Q) \rightarrow R$

1. $(P \rightarrow Q) \rightarrow R$

2. $P \leftrightarrow Q$ for $\rightarrow$Intro

3. P for $\rightarrow$Intro

4. Q $\leftrightarrow$Elim 2,3

$P \rightarrow Q$ $\rightarrow$Intro 3-

R $\rightarrow$Elim 1,

$(P \leftrightarrow Q) \rightarrow R$ $\rightarrow$Intro 2-

Think Backwards

Example:

$(P \rightarrow Q) \rightarrow R$

$(P \leftrightarrow Q) \rightarrow R$

1. $(P \rightarrow Q) \rightarrow R$

2. $P \leftrightarrow Q$ for $\rightarrow$Intro

3. P for $\rightarrow$Intro

4. Q $\leftrightarrow$Elim 2,3

5. $P \rightarrow Q$ $\rightarrow$Intro 3-4

R $\rightarrow$Elim 1,

$(P \leftrightarrow Q) \rightarrow R$ $\rightarrow$Intro 2-

Think Backwards

Example:

$(P \to Q) \to R$

$(P \leftrightarrow Q) \to R$

1. $(P \to Q) \to R$
2. $P \leftrightarrow Q$ for $\to$Intro
3. P for $\to$Intro
4. Q $\leftrightarrow$Elim 2,3
5. $P \to Q$ $\to$Intro 3-4
6. R $\to$Elim 1,5

$(P \leftrightarrow Q) \to R$ $\to$Intro 2-

Think Backwards

Example:

$(P \to Q) \to R$

$(P \leftrightarrow Q) \to R$

1.	$(P \to Q) \to R$	
2.	$P \leftrightarrow Q$	for $\to$Intro
3.	P	for $\to$Intro
4.	Q	$\leftrightarrow$Elim 2,3
5.	$P \to Q$	$\to$Intro 3-4
6.	R	$\to$Elim 1,5
7.	$(P \leftrightarrow Q) \to R$	$\to$Intro 2-6

Think Backwards

Example:

$(P \rightarrow Q) \rightarrow R$

$S \leftrightarrow Q$

$(P \rightarrow S) \rightarrow R$

1. $(P \rightarrow Q) \rightarrow R$
2. $S \leftrightarrow Q$

$(P \rightarrow S) \rightarrow R$

THINK BACKWARDS

Example:

| (P → Q) → R
| S ↔ Q
|———
| (P → S) → R

| 1. (P → Q) → R
| 2. S ↔ Q
|———
| | 3. P → S   for → Intro
| |———
| |
| | R
| (P → S) → R   → Intro 3-

Think Backwards

Example:

$(P \rightarrow Q) \rightarrow R$

$S \leftrightarrow Q$

$(P \rightarrow S) \rightarrow R$

1. $(P \rightarrow Q) \rightarrow R$
2. $S \leftrightarrow Q$
3. $P \rightarrow S$ for $\rightarrow$Intro

How to get R?

R

$(P \rightarrow S) \rightarrow R$ $\rightarrow$Intro 3-

THINK BACKWARDS

Example:

$(P \rightarrow Q) \rightarrow R$

$S \leftrightarrow Q$

$(P \rightarrow S) \rightarrow R$

1. $(P \rightarrow Q) \rightarrow R$
2. $S \leftrightarrow Q$

3. $P \rightarrow S$ — for →Intro

How to get R?
From line 1

R

$(P \rightarrow S) \rightarrow R$ — →Intro 3-

Think Backwards

Example:

$(P \rightarrow Q) \rightarrow R$

$S \leftrightarrow Q$

$(P \rightarrow S) \rightarrow R$

1. $(P \rightarrow Q) \rightarrow R$
2. $S \leftrightarrow Q$

---

3. $P \rightarrow S$ — for $\rightarrow$Intro

How to get R?
From line 1

$P \rightarrow Q$

R — $\rightarrow$Elim 1,

$(P \rightarrow S) \rightarrow R$ — $\rightarrow$Intro 3-

1. $(P \rightarrow Q) \rightarrow R$

2. $S \leftrightarrow Q$

---

3. $P \rightarrow S$ for $\rightarrow$Intro

$P \rightarrow Q$

R $\rightarrow$Elim 1,

$(P \rightarrow S) \rightarrow R$ $\rightarrow$Intro 3-

1. $(P \to Q) \to R$

2. $S \leftrightarrow Q$

---

3. $P \to S$ — for $\to$Intro

---

4. P — for $\to$Intro

Q

$P \to Q$ — $\to$Intro 4-

R — $\to$Elim 1,

$(P \to S) \to R$ — $\to$Intro 3-

1. (P → Q) → R

2. S ↔ Q

---

3. P → S — for → Intro

---

4. P — for → Intro

---

5. S — → Elim 3,4

Q

P → Q — → Intro 4-

R — → Elim 1,

(P → S) → R — → Intro 3-

1. (P → Q) → R

2. S ↔ Q

3. P → S for → Intro

4. P for → Intro

5. S → Elim 3,4

6. Q ↔ Elim 2,5

Q

P → Q → Intro 4-

R → Elim 1,

(P → S) → R → Intro 3-

1. (P → Q) → R
2. S ↔ Q

3. P → S — for → Intro

4. P — for → Intro

5. S — → Elim 3,4

6. Q — ↔ Elim 2,5

7. P → Q — → Intro 4-6

R — → Elim 1,

(P → S) → R — → Intro 3-

1. $(P \rightarrow Q) \rightarrow R$

2. $S \leftrightarrow Q$

3. $P \rightarrow S$ for $\rightarrow$Intro

4. P for $\rightarrow$Intro

5. S $\rightarrow$Elim 3,4

6. Q $\leftrightarrow$Elim 2,5

7. $P \rightarrow Q$ $\rightarrow$Intro 4-6

8. R $\rightarrow$Elim 1,7

$(P \rightarrow S) \rightarrow R$ $\rightarrow$Intro 3-

1. $(P \rightarrow Q) \rightarrow R$

2. $S \leftrightarrow Q$

3. $P \rightarrow S$ for $\rightarrow$Intro

4. P for $\rightarrow$Intro

5. S $\rightarrow$Elim 3,4

6. Q $\leftrightarrow$Elim 2,5

7. $P \rightarrow Q$ $\rightarrow$Intro 4-6

8. R $\rightarrow$Elim 1,7

9. $(P \rightarrow S) \rightarrow R$ $\rightarrow$Intro 3-8

Example

Example:

$P \leftrightarrow Q$

$Q \leftrightarrow R$

$(P \rightarrow R) \wedge (R \rightarrow P)$

1. $P \leftrightarrow Q$
2. $Q \leftrightarrow R$

$(P \rightarrow R) \wedge (R \rightarrow P)$

Example

Example:

$P \leftrightarrow Q$

$Q \leftrightarrow R$

$(P \rightarrow R) \land (R \rightarrow P)$

1. $P \leftrightarrow Q$
2. $Q \leftrightarrow R$

---

3. P   for $\rightarrow$Intro

$(P \rightarrow R) \land (R \rightarrow P)$

Example

Example:

$P \leftrightarrow Q$

$Q \leftrightarrow R$

$(P \rightarrow R) \wedge (R \rightarrow P)$

1. $P \leftrightarrow Q$
2. $Q \leftrightarrow R$

---

3. P — for $\rightarrow$Intro

---

4. Q — $\leftrightarrow$Elim 1,3

$(P \rightarrow R) \wedge (R \rightarrow P)$

Example

Example:

$P \leftrightarrow Q$

$Q \leftrightarrow R$

$(P \rightarrow R) \wedge (R \rightarrow P)$

1. $P \leftrightarrow Q$
2. $Q \leftrightarrow R$

---

3. P for $\rightarrow$Intro

---

4. Q $\leftrightarrow$Elim 1,3
5. R $\leftrightarrow$Elim 2,4

$(P \rightarrow R) \wedge (R \rightarrow P)$

EXAMPLE

Example:

$P \leftrightarrow Q$

$Q \leftrightarrow R$

$(P \rightarrow R) \wedge (R \rightarrow P)$

1. $P \leftrightarrow Q$
2. $Q \leftrightarrow R$

3.	P	for $\rightarrow$Intro
4.	Q	$\leftrightarrow$Elim 1,3
5.	R	$\leftrightarrow$Elim 2,4
6.	$P \rightarrow R$	$\rightarrow$Intro 3-5

$(P \rightarrow R) \wedge (R \rightarrow P)$

Example

Example:

$P \leftrightarrow Q$

$Q \leftrightarrow R$

$(P \rightarrow R) \wedge (R \rightarrow P)$

1. $P \leftrightarrow Q$
2. $Q \leftrightarrow R$
3. P — for $\rightarrow$Intro
4. Q — $\leftrightarrow$Elim 1,3
5. R — $\leftrightarrow$Elim 2,4
6. $P \rightarrow R$ — $\rightarrow$Intro 3-5
7. R — for $\rightarrow$Intro

$(P \rightarrow R) \wedge (R \rightarrow P)$

Example

Example:

$P \leftrightarrow Q$

$Q \leftrightarrow R$

$(P \rightarrow R) \wedge (R \rightarrow P)$

1. $P \leftrightarrow Q$
2. $Q \leftrightarrow R$
3. P for $\rightarrow$Intro
4. Q $\leftrightarrow$Elim 1,3
5. R $\leftrightarrow$Elim 2,4
6. $P \rightarrow R$ $\rightarrow$Intro 3-5
7. R for $\rightarrow$Intro
8. Q $\leftrightarrow$Elim 2,7

$(P \rightarrow R) \wedge (R \rightarrow P)$

Example

Example:

$P \leftrightarrow Q$

$Q \leftrightarrow R$

$(P \rightarrow R) \wedge (R \rightarrow P)$

1. $P \leftrightarrow Q$
2. $Q \leftrightarrow R$

---

3. P — for $\rightarrow$Intro
4. Q — $\leftrightarrow$Elim 1,3
5. R — $\leftrightarrow$Elim 2,4
6. $P \rightarrow R$ — $\rightarrow$Intro 3-5
7. R — for $\rightarrow$Intro
8. Q — $\leftrightarrow$Elim 2,7
9. P — $\leftrightarrow$Elim 1,8

$(P \rightarrow R) \wedge (R \rightarrow P)$

Example

Example:

$P \leftrightarrow Q$

$Q \leftrightarrow R$

$(P \rightarrow R) \land (R \rightarrow P)$

1. $P \leftrightarrow Q$
2. $Q \leftrightarrow R$
3. P — for $\rightarrow$Intro
4. Q — $\leftrightarrow$Elim 1,3
5. R — $\leftrightarrow$Elim 2,4
6. $P \rightarrow R$ — $\rightarrow$Intro 3-5
7. R — for $\rightarrow$Intro
8. Q — $\leftrightarrow$Elim 2,7
9. P — $\leftrightarrow$Elim 1,8
10. $R \rightarrow P$ — $\rightarrow$Intro 7-9

$(P \rightarrow R) \land (R \rightarrow P)$

Example

Example:

$P \leftrightarrow Q$

$Q \leftrightarrow R$

$(P \rightarrow R) \wedge (R \rightarrow P)$

1. $P \leftrightarrow Q$
2. $Q \leftrightarrow R$

---

3. P for $\rightarrow$Intro

---

4. Q $\leftrightarrow$Elim 1,3
5. R $\leftrightarrow$Elim 2,4
6. $P \rightarrow R$ $\rightarrow$Intro 3-5
7. R for $\rightarrow$Intro

---

8. Q $\leftrightarrow$Elim 2,7
9. P $\leftrightarrow$Elim 1,8
10. $R \rightarrow P$ $\rightarrow$Intro 7-9
11. $(P \rightarrow R) \wedge (R \rightarrow P)$ $\wedge$Intro 6,10

Formal Proof Rules

- ↔ Introduction: from a proof from P to Q and a proof from Q to P, we can infer P ↔ Q.

```
|  | 1. P
|  |----
|  | ...
|  | j. Q
|  | k. Q
|  |----
|  | ...
|  | m. P
|
|  k. P ↔ Q          ↔ Intro: 1-j, k-m
```

Formal Proof Rules

- ↔ Introduction: from a proof from P to Q and a proof from Q to P, we can infer P ↔ Q.

```
| | 1. P
| |----
| | ...
| | j. Q
| | k. Q
| |----
| | ...
| | m. P
| k. P ↔ Q        ↔ Intro: 1-j, k-m
```

Proving a biconditional just is proving two conditionals

Biconditional Introduction

Example:

$P \leftrightarrow Q$

$Q \leftrightarrow R$

$P \leftrightarrow R$

1. $P \leftrightarrow Q$
2. $Q \leftrightarrow R$

$P \leftrightarrow R$

Biconditional Introduction

Example:

$P \leftrightarrow Q$

$Q \leftrightarrow R$

$P \leftrightarrow R$

1. $P \leftrightarrow Q$
2. $Q \leftrightarrow R$

---

3. P   for ↔Intro

$P \leftrightarrow R$

Biconditional Introduction

Example:

P↔Q

Q↔R

P↔R

1. P↔Q
2. Q↔R

---

3. P   for ↔Intro

---

4. Q   ↔Elim 1,3

P↔R

Biconditional Introduction

Example:

P↔Q

Q↔R

P↔R

1. P↔Q
2. Q↔R

---

3. P — for ↔Intro

---

4. Q — ↔Elim 1,3
5. R — ↔Elim 2,4

P↔R

Biconditional Introduction

Example:

$P \leftrightarrow Q$

$Q \leftrightarrow R$

$P \leftrightarrow R$

1. $P \leftrightarrow Q$
2. $Q \leftrightarrow R$

---

3. P — for ↔Intro

---

4. Q — ↔Elim 1,3
5. R — ↔Elim 2,4

6. R — for ↔Intro

$P \leftrightarrow R$

Biconditional Introduction

Example:

| P↔Q
| Q↔R
|———
| P↔R

| 1. P↔Q |
| 2. Q↔R |
|———
| | 3. P | for ↔Intro |
| |———
| | 4. Q | ↔Elim 1,3 |
| | 5. R | ↔Elim 2,4 |

| | 6. R | for ↔Intro |
| |———
| | 7. Q | ↔Elim 2,6 |

| P↔R

Biconditional Introduction

Example:

| P↔Q
| Q↔R
|---
| P↔R

1. P↔Q
2. Q↔R

---

3. P for ↔Intro
4. Q ↔Elim 1,3
5. R ↔Elim 2,4

6. R for ↔Intro
7. Q ↔Elim 2,6
8. P ↔Elim 1,7

P↔R

Biconditional Introduction

Example:

$$\begin{array}{l|l} & P \leftrightarrow Q \\ & Q \leftrightarrow R \\ \hline & P \leftrightarrow R \end{array}$$

1. P↔Q
2. Q↔R
   3. P — for ↔Intro
   4. Q — ↔Elim 1,3
   5. R — ↔Elim 2,4

   6. R — for ↔Intro
   7. Q — ↔Elim 2,6
   8. P — ↔Elim 1,7
9. P↔R — ↔Intro 3-5, 6-8

1. P↔Q
2. Q↔R
3. P — for →Intro
4. Q — ↔Elim 1,3
5. R — ↔Elim 2,4
6. P→R — →Intro 3-5
7. R — for →Intro
8. Q — ↔Elim 2,7
9. P — ↔Elim 1,8
10. R→P — →Intro 7-9
11. (P→R)∧(R→P) — ∧Intro 6,10

1. P↔Q
2. Q↔R
3. P — for ↔Intro
4. Q — ↔Elim 1,3
5. R — ↔Elim 2,4
6. R — for ↔Intro
7. Q — ↔Elim 2,6
8. P — ↔Elim 1,7
9. P↔R — ↔Intro 3-5, 6-8

Biconditional Introduction

Example:

P↔Q

¬P↔¬Q

Biconditional Introduction

Example:

$P \leftrightarrow Q$

$\neg P \leftrightarrow \neg Q$

1. $P \leftrightarrow Q$

$\neg P \leftrightarrow \neg Q$

Biconditional Introduction

Example:

| P↔Q
|---
| ¬P↔¬Q

1. P↔Q

---

    2. ¬P — for ↔Intro

    ¬Q

    ¬Q — for ↔Intro

    ¬P

¬P↔¬Q — ↔Intro

1. $P \leftrightarrow Q$

$\neg P \leftrightarrow \neg Q$

1. P↔Q

2. ¬P   for ↔Intro

¬Q

¬P↔¬Q   ↔Intro

1. P↔Q

2. ¬P — for ↔Intro

3. Q — for ¬Intro

¬Q — ¬Intro

¬P↔¬Q — ↔Intro

1. P↔Q	
2. ¬P	for ↔Intro
3. Q	for ¬Intro
4. P	↔Elim 1,3
¬Q	¬Intro
¬P↔¬Q	↔Intro

1. P↔Q	
2. ¬P	for ↔Intro
3. Q	for ¬Intro
4. P	↔Elim 1,3
5. ⊥	⊥ Intro 2,4
¬Q	¬Intro
¬P↔¬Q	↔Intro

1. P↔Q	
2. ¬P	for ↔Intro
3. Q	for ¬Intro
4. P	↔Elim 1,3
5. ⊥	⊥ Intro 2,4
6. ¬Q	¬Intro 3-5
¬P↔¬Q	↔Intro

1. P↔Q

2. ¬P — for ↔Intro

3. Q — for ¬Intro

4. P — ↔Elim 1,3

5. ⊥ — ⊥ Intro 2,4

6. ¬Q — ¬Intro 3-5

7. ¬Q — for ↔Intro

¬P

¬P↔¬Q — ↔Intro

1. P↔Q	
2. ¬P	for ↔Intro
3. Q	for ¬Intro
4. P	↔Elim 1,3
5. ⊥	⊥ Intro 2,4
6. ¬Q	¬Intro 3-5
7. ¬Q	for ↔Intro
8. P	for ¬Intro
¬P	
¬P↔¬Q	↔Intro

1. P↔Q		
2. ¬P	for ↔Intro	
3. Q	for ¬Intro	
4. P	↔Elim 1,3	
5. ⊥	⊥ Intro 2,4	
6. ¬Q	¬Intro 3-5	
7. ¬Q	for ↔Intro	
8. P	for ¬Intro	
9. Q	↔Elim 1,8	
¬P		
¬P↔¬Q	↔Intro	

1. $P \leftrightarrow Q$

2. $\neg P$ — for $\leftrightarrow$Intro

3. Q — for $\neg$Intro

4. P — $\leftrightarrow$Elim 1,3

5. $\bot$ — $\bot$ Intro 2,4

6. $\neg Q$ — $\neg$Intro 3-5

7. $\neg Q$ — for $\leftrightarrow$Intro

8. P — for $\neg$Intro

9. Q — $\leftrightarrow$Elim 1,8

10. $\bot$ — $\bot$ Intro 7,9

$\neg P$

$\neg P \leftrightarrow \neg Q$ — $\leftrightarrow$Intro

1. $P \leftrightarrow Q$	
2. $\neg P$	for $\leftrightarrow$Intro
3. Q	for $\neg$Intro
4. P	$\leftrightarrow$Elim 1,3
5. $\bot$	$\bot$ Intro 2,4
6. $\neg Q$	$\neg$Intro 3-5
7. $\neg Q$	for $\leftrightarrow$Intro
8. P	for $\neg$Intro
9. Q	$\leftrightarrow$Elim 1,8
10. $\bot$	$\bot$ Intro 7,9
11. $\neg P$	$\neg$Intro 8-10
$\neg P \leftrightarrow \neg Q$	$\leftrightarrow$Intro

1. $P \leftrightarrow Q$

2. $\neg P$ — for $\leftrightarrow$Intro

3. Q — for $\neg$Intro

4. P — $\leftrightarrow$Elim 1,3

5. $\bot$ — $\bot$ Intro 2,4

6. $\neg Q$ — $\neg$Intro 3-5

7. $\neg Q$ — for $\leftrightarrow$Intro

8. P — for $\neg$Intro

9. Q — $\leftrightarrow$Elim 1,8

10. $\bot$ — $\bot$ Intro 7,9

11. $\neg P$ — $\neg$Intro 8-10

12. $\neg P \leftrightarrow \neg Q$ — $\leftrightarrow$Intro 2-6, 7-11

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- If two sentences are equivalent, then you could do a proof from the first to the second and you could also do a proof from the second to the first.

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    - ---- So you can prove the biconditional from no premises at all (by doing the two relevant proofs and then sticking them together

EXAMPLES OF EQUIVALENCES

DeMorgan's Laws

$$\begin{array}{l|l} & \neg(P \lor Q) \\ \hline & \neg P \land \neg Q \end{array}$$

and also

$$\begin{array}{l|l} & \neg P \land \neg Q \\ \hline & \neg(P \lor Q) \end{array}$$

so by doing both proofs and then doing ↔Intro

$$\vdash \neg(P \lor Q) \leftrightarrow (\neg P \land \neg Q)$$

Examples of Equivalences

Contraposition

| $P \to Q$
|———
| $\neg Q \to \neg P$

and also

| $\neg Q \to \neg P$
|———
| $P \to Q$

so by doing both proofs and then doing $\leftrightarrow$Intro

|
|———
| $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$