

WHICH ARE PROVABLE?

$$\begin{array}{|l} P \rightarrow R \\ \hline (P \wedge Q) \rightarrow R \end{array}$$

$$\begin{array}{|l} P \rightarrow R \\ \hline (P \vee Q) \rightarrow R \end{array}$$

$$\begin{array}{|l} (P \wedge Q) \rightarrow R \\ \hline P \rightarrow R \end{array}$$

$$\begin{array}{|l} (P \vee Q) \rightarrow R \\ \hline P \rightarrow R \end{array}$$

WHICH ARE PROVABLE?

$$\begin{array}{|l} P \rightarrow R \\ \hline (P \wedge Q) \rightarrow R \end{array} \quad \text{VALID}$$

$$\begin{array}{|l} P \rightarrow R \\ \hline (P \vee Q) \rightarrow R \end{array} \quad \text{INVALID}$$

$$\begin{array}{|l} (P \wedge Q) \rightarrow R \\ \hline P \rightarrow R \end{array} \quad \text{INVALID}$$

$$\begin{array}{|l} (P \vee Q) \rightarrow R \\ \hline P \rightarrow R \end{array} \quad \text{VALID}$$

PROOFS WITH CONDITIONALS III

Friday, 28 February

RULES FOR CONDITIONALS

- \rightarrow Elimination: from $P \rightarrow Q$ and P , we can infer Q .

$$\begin{array}{l|l} 1. P \rightarrow Q & \\ 2. P & \\ \hline 3. Q & \rightarrow \text{Elim: 1,2} \end{array}$$

- \leftrightarrow Elimination: from $P \leftrightarrow Q$ and P/Q , we can infer Q/P .

$$\begin{array}{l|l} 1. P \leftrightarrow Q & \\ 2. Q & \\ \hline 3. P & \leftrightarrow \text{Elim: 1,2} \end{array}$$

FORMAL PROOF RULES

- \rightarrow Introduction

From a proof from P to Q , we can infer $P \rightarrow Q$.



This rule is often known as **Conditional Proof**

THINK MAIN CONNECTIVE

Example:

$$P \rightarrow T$$

$$S \leftrightarrow T$$

$$(S \leftrightarrow R) \rightarrow (P \rightarrow R)$$

THINK MAIN CONNECTIVE

Example:

$$P \rightarrow T$$

$$S \leftrightarrow T$$

$$(S \leftrightarrow R) \rightarrow (P \rightarrow R))$$

$$1. P \rightarrow T$$

$$2. S \leftrightarrow T$$

$$(S \leftrightarrow R) \rightarrow (P \rightarrow R))$$

THINK MAIN CONNECTIVE

Example:

$$\frac{\begin{array}{l} P \rightarrow T \\ S \leftrightarrow T \end{array}}{(S \leftrightarrow R) \rightarrow (P \rightarrow R)}$$

$$1. P \rightarrow T$$

$$2. S \leftrightarrow T$$

$$3. S \leftrightarrow R$$

for \rightarrow Intro

$$P \rightarrow R$$

$$(S \leftrightarrow R) \rightarrow (P \rightarrow R) \quad \text{by } \rightarrow \text{Intro 3-}$$

THINK MAIN CONNECTIVE

Example:

$$\begin{array}{|l} P \rightarrow T \\ S \leftrightarrow T \\ \hline (S \leftrightarrow R) \rightarrow (P \rightarrow R) \end{array}$$

$$1. P \rightarrow T$$

$$2. S \leftrightarrow T$$

$$3. S \leftrightarrow R$$

for \rightarrow Intro

$$4. P$$

for \rightarrow Intro

$$R$$

$$P \rightarrow R$$

by \rightarrow Intro 4-

$$(S \leftrightarrow R) \rightarrow (P \rightarrow R)$$

by \rightarrow Intro 3-

THINK MAIN CONNECTIVE

Example:

$$\frac{\begin{array}{l} P \rightarrow T \\ S \leftrightarrow T \end{array}}{(S \leftrightarrow R) \rightarrow (P \rightarrow R)}$$

$$1. P \rightarrow T$$

$$2. S \leftrightarrow T$$

$$3. S \leftrightarrow R$$

for \rightarrow Intro

$$4. P$$

for \rightarrow Intro

$$5. T$$

\rightarrow Elim 1,5

$$R$$

$$P \rightarrow R$$

by \rightarrow Intro 4-

$$(S \leftrightarrow R) \rightarrow (P \rightarrow R)$$

by \rightarrow Intro 3-

THINK MAIN CONNECTIVE

Example:

$$\frac{\begin{array}{l} P \rightarrow T \\ S \leftrightarrow T \end{array}}{(S \leftrightarrow R) \rightarrow (P \rightarrow R)}$$

$$1. P \rightarrow T$$

$$2. S \leftrightarrow T$$

$$3. S \leftrightarrow R$$

for \rightarrow Intro

$$4. P$$

for \rightarrow Intro

$$5. T$$

\rightarrow Elim 1,5

$$6. S$$

\leftrightarrow Elim 2,5

$$R$$

$$P \rightarrow R$$

by \rightarrow Intro 4-

$$(S \leftrightarrow R) \rightarrow (P \rightarrow R)$$

by \rightarrow Intro 3-

THINK MAIN CONNECTIVE

Example:

$$\frac{\begin{array}{l} P \rightarrow T \\ S \leftrightarrow T \end{array}}{(S \leftrightarrow R) \rightarrow (P \rightarrow R)}$$

$$1. P \rightarrow T$$

$$2. S \leftrightarrow T$$

$$3. S \leftrightarrow R$$

for \rightarrow Intro

$$4. P$$

for \rightarrow Intro

$$5. T$$

\rightarrow Elim 1,5

$$6. S$$

\leftrightarrow Elim 2,5

$$7. R$$

\leftrightarrow Elim 3,6

$$P \rightarrow R$$

by \rightarrow Intro 4-

$$(S \leftrightarrow R) \rightarrow (P \rightarrow R)$$

by \rightarrow Intro 3-

THINK MAIN CONNECTIVE

Example:

$$\frac{\begin{array}{l} P \rightarrow T \\ S \leftrightarrow T \end{array}}{(S \leftrightarrow R) \rightarrow (P \rightarrow R)}$$

$$1. P \rightarrow T$$

$$2. S \leftrightarrow T$$

$$3. S \leftrightarrow R$$

for \rightarrow Intro

$$4. P$$

for \rightarrow Intro

$$5. T$$

\rightarrow Elim 1,4

$$6. S$$

\leftrightarrow Elim 2,5

$$7. R$$

\leftrightarrow Elim 3,6

$$8. P \rightarrow R$$

\rightarrow Intro 4-7

$$(S \leftrightarrow R) \rightarrow (P \rightarrow R)$$

by \rightarrow Intro 3-

THINK MAIN CONNECTIVE

Example:

$$\frac{\begin{array}{l} P \rightarrow T \\ S \leftrightarrow T \end{array}}{(S \leftrightarrow R) \rightarrow (P \rightarrow R)}$$

$$1. P \rightarrow T$$

$$2. S \leftrightarrow T$$

$$3. S \leftrightarrow R$$

for \rightarrow Intro

$$4. P$$

for \rightarrow Intro

$$5. T$$

\rightarrow Elim 1,4

$$6. S$$

\leftrightarrow Elim 2,5

$$7. R$$

\leftrightarrow Elim 3,6

$$8. P \rightarrow R$$

\rightarrow Intro 4-7

$$9. (S \leftrightarrow R) \rightarrow (P \rightarrow R) \rightarrow \text{Intro 3-8}$$

THINK BACKWARDS

Example:

$$\begin{array}{l} (P \rightarrow Q) \rightarrow R \\ \hline (P \leftrightarrow Q) \rightarrow R \end{array}$$

$$1. (P \rightarrow Q) \rightarrow R$$

$$(P \leftrightarrow Q) \rightarrow R$$

THINK BACKWARDS

Example:

$$\frac{(P \rightarrow Q) \rightarrow R}{(P \leftrightarrow Q) \rightarrow R}$$

$$1. (P \rightarrow Q) \rightarrow R$$

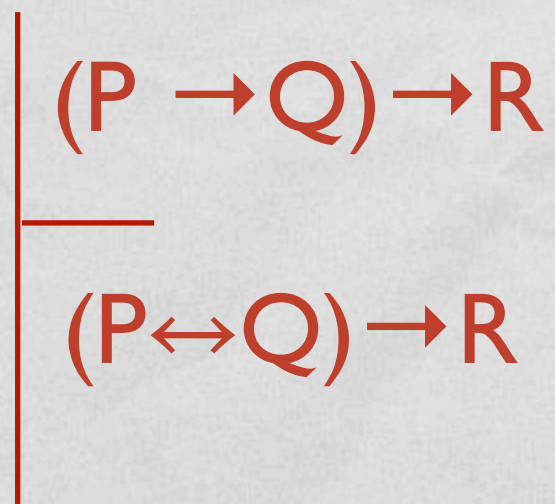
$$2. P \leftrightarrow Q \quad \text{for } \rightarrow \text{Intro}$$

R

$$(P \leftrightarrow Q) \rightarrow R \quad \rightarrow \text{Intro 2-}$$

THINK BACKWARDS

Example:



1. $(P \rightarrow Q) \rightarrow R$

2. $P \leftrightarrow Q$ for \rightarrow Intro

How to get R?

R

$(P \leftrightarrow Q) \rightarrow R$ \rightarrow Intro 2-

THINK BACKWARDS

Example:

$$\frac{(P \rightarrow Q) \rightarrow R}{(P \leftrightarrow Q) \rightarrow R}$$

$$1. (P \rightarrow Q) \rightarrow R$$

$$2. P \leftrightarrow Q \quad \text{for } \rightarrow \text{Intro}$$

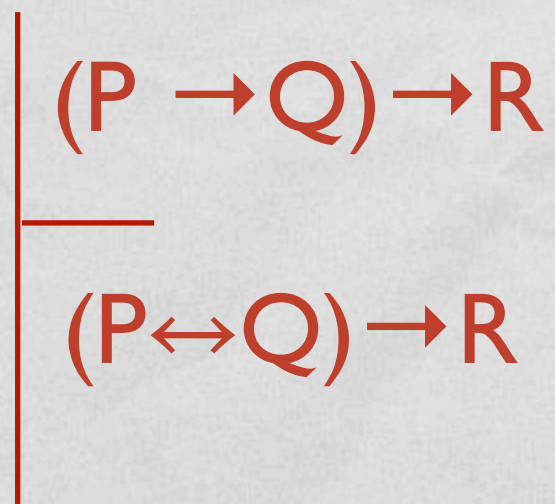
How to get R?
From line 1

R

$$(P \leftrightarrow Q) \rightarrow R \quad \rightarrow \text{Intro 2-}$$

THINK BACKWARDS

Example:



1. $(P \rightarrow Q) \rightarrow R$

2. $P \leftrightarrow Q$ for \rightarrow Intro

How to get R?
From line 1

$P \rightarrow Q$

R

\rightarrow Elim 1,

$(P \leftrightarrow Q) \rightarrow R$ \rightarrow Intro 2-

THINK BACKWARDS

Example:

$$\frac{(P \rightarrow Q) \rightarrow R}{(P \leftrightarrow Q) \rightarrow R}$$

$$1. (P \rightarrow Q) \rightarrow R$$

$$2. P \leftrightarrow Q \quad \text{for } \rightarrow \text{Intro}$$

$$P \rightarrow Q$$

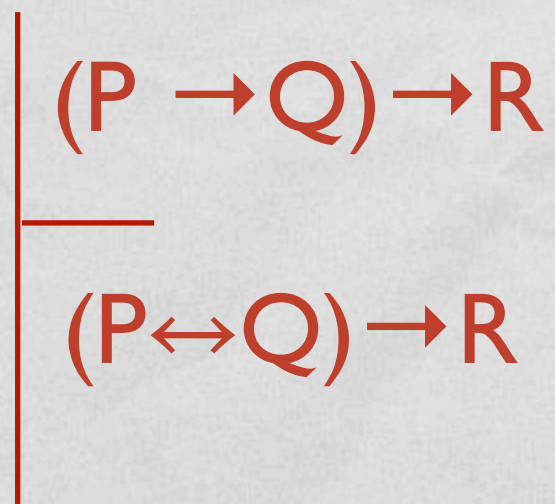
$$R$$

\rightarrow Elim 1,

$$(P \leftrightarrow Q) \rightarrow R \quad \rightarrow \text{Intro 2-}$$

THINK BACKWARDS

Example:



1. $(P \rightarrow Q) \rightarrow R$

2. $P \leftrightarrow Q$ for \rightarrow Intro

3. P for \rightarrow Intro

Q

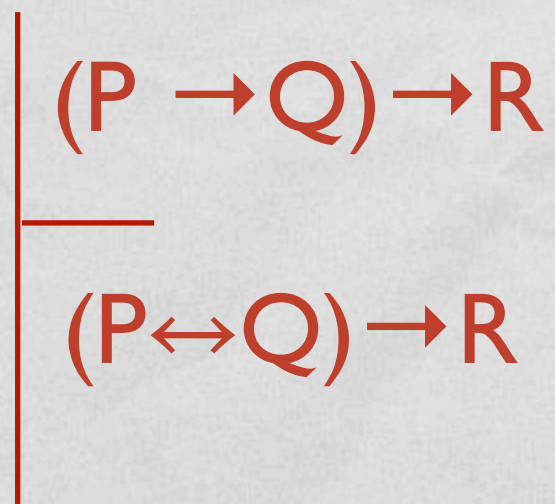
$P \rightarrow Q$ \rightarrow Intro 3-

R \rightarrow Elim 1,

$(P \leftrightarrow Q) \rightarrow R$ \rightarrow Intro 2-

THINK BACKWARDS

Example:



1. $(P \rightarrow Q) \rightarrow R$

2. $P \leftrightarrow Q$ for \rightarrow Intro

3. P for \rightarrow Intro

4. Q \leftrightarrow Elim 2,3

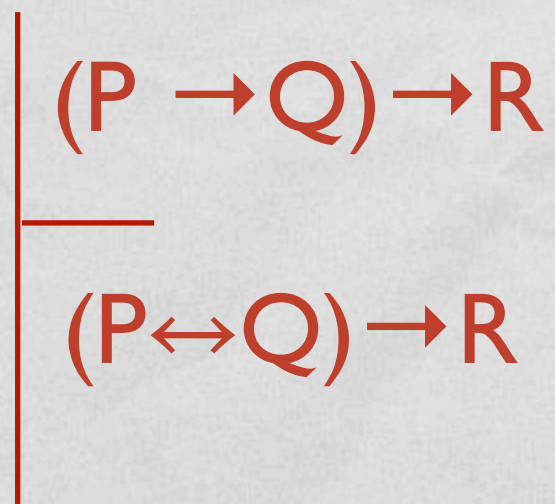
$P \rightarrow Q$ \rightarrow Intro 3-

R \rightarrow Elim 1,

$(P \leftrightarrow Q) \rightarrow R$ \rightarrow Intro 2-

THINK BACKWARDS

Example:



1. $(P \rightarrow Q) \rightarrow R$

2. $P \leftrightarrow Q$ for \rightarrow Intro

3. P for \rightarrow Intro

4. Q \leftrightarrow Elim 2,3

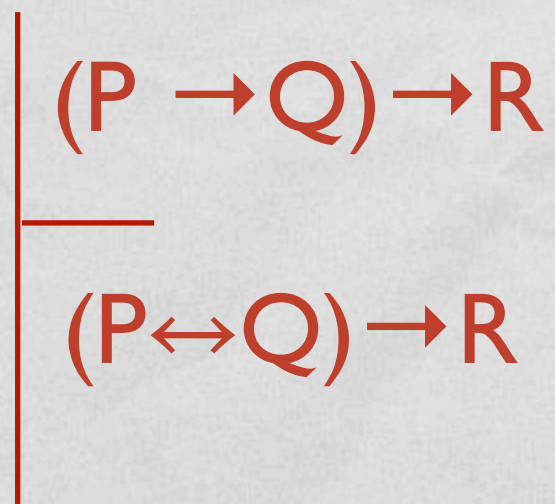
5. $P \rightarrow Q$ \rightarrow Intro 3-4

R \rightarrow Elim 1,

$(P \leftrightarrow Q) \rightarrow R$ \rightarrow Intro 2-

THINK BACKWARDS

Example:



1. $(P \rightarrow Q) \rightarrow R$

2. $P \leftrightarrow Q$ for \rightarrow Intro

3. P for \rightarrow Intro

4. Q \leftrightarrow Elim 2,3

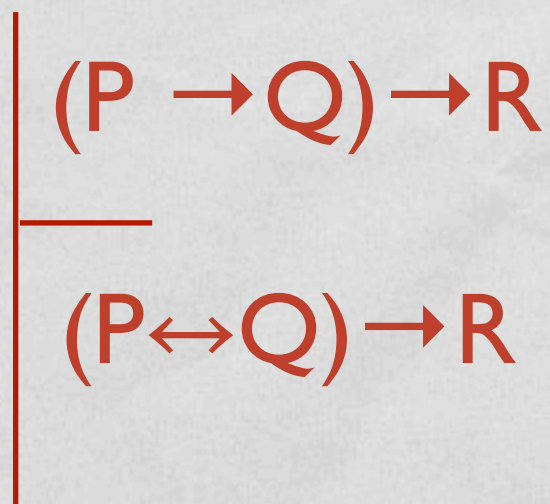
5. $P \rightarrow Q$ \rightarrow Intro 3-4

6. R \rightarrow Elim 1,5

$(P \leftrightarrow Q) \rightarrow R$ \rightarrow Intro 2-6

THINK BACKWARDS

Example:



1. $(P \rightarrow Q) \rightarrow R$

2. $P \leftrightarrow Q$ for \rightarrow Intro

3. P for \rightarrow Intro

4. Q \leftrightarrow Elim 2,3

5. $P \rightarrow Q$ \rightarrow Intro 3-4

6. R \rightarrow Elim 1,5

7. $(P \leftrightarrow Q) \rightarrow R$ \rightarrow Intro 2-6

THINK BACKWARDS

Example:

$$(P \rightarrow Q) \rightarrow R$$

$$S \leftrightarrow Q$$

$$(P \rightarrow S) \rightarrow R$$

$$1. (P \rightarrow Q) \rightarrow R$$

$$2. S \leftrightarrow Q$$

$$(P \rightarrow S) \rightarrow R$$

THINK BACKWARDS

Example:

$$\frac{\begin{array}{l} (P \rightarrow Q) \rightarrow R \\ S \leftrightarrow Q \end{array}}{(P \rightarrow S) \rightarrow R}$$

1. $(P \rightarrow Q) \rightarrow R$

2. $S \leftrightarrow Q$

3. $P \rightarrow S$ for \rightarrow Intro

R
 $(P \rightarrow S) \rightarrow R$ \rightarrow Intro 3-

THINK BACKWARDS

Example:

$$\begin{array}{|l} (P \rightarrow Q) \rightarrow R \\ S \leftrightarrow Q \\ \hline (P \rightarrow S) \rightarrow R \end{array}$$

$$1. (P \rightarrow Q) \rightarrow R$$

$$2. S \leftrightarrow Q$$

$$3. P \rightarrow S \quad \text{for } \rightarrow \text{Intro}$$

How to get R?

$$\begin{array}{|l} R \\ (P \rightarrow S) \rightarrow R \quad \rightarrow \text{Intro 3-} \end{array}$$

THINK BACKWARDS

Example:

$$\begin{array}{|l} (P \rightarrow Q) \rightarrow R \\ S \leftrightarrow Q \\ \hline (P \rightarrow S) \rightarrow R \end{array}$$

1. $(P \rightarrow Q) \rightarrow R$

2. $S \leftrightarrow Q$

3. $P \rightarrow S$ for \rightarrow Intro

How to get R?
From line 1

R
 $(P \rightarrow S) \rightarrow R$ \rightarrow Intro 3-

THINK BACKWARDS

Example:

$$\begin{array}{|l} (P \rightarrow Q) \rightarrow R \\ S \leftrightarrow Q \\ \hline (P \rightarrow S) \rightarrow R \end{array}$$

1. $(P \rightarrow Q) \rightarrow R$

2. $S \leftrightarrow Q$

3. $P \rightarrow S$ for \rightarrow Intro

How to get R?
From line 1

$P \rightarrow Q$

R

$(P \rightarrow S) \rightarrow R$

\rightarrow Elim 1,
 \rightarrow Intro 3-

1. $(P \rightarrow Q) \rightarrow R$

2. $S \leftrightarrow Q$

3. $P \rightarrow S$ for \rightarrow Intro

$P \rightarrow Q$

R

$(P \rightarrow S) \rightarrow R$

\rightarrow Elim 1,

\rightarrow Intro 3-

1. $(P \rightarrow Q) \rightarrow R$

2. $S \leftrightarrow Q$

3. $P \rightarrow S$

for \rightarrow Intro

4. P

for \rightarrow Intro

Q

$P \rightarrow Q$

\rightarrow Intro 4-

R

\rightarrow Elim 1,

$(P \rightarrow S) \rightarrow R$

\rightarrow Intro 3-

1. $(P \rightarrow Q) \rightarrow R$

2. $S \leftrightarrow Q$

3. $P \rightarrow S$

for \rightarrow Intro

4. P

for \rightarrow Intro

5. S

\rightarrow Elim 3,4

Q

$P \rightarrow Q$

\rightarrow Intro 4-

R

\rightarrow Elim 1,

$(P \rightarrow S) \rightarrow R$

\rightarrow Intro 3-

1. $(P \rightarrow Q) \rightarrow R$

2. $S \leftrightarrow Q$

3. $P \rightarrow S$ for \rightarrow Intro

4. P for \rightarrow Intro

5. S \rightarrow Elim 3,4

6. Q \leftrightarrow Elim 2,5

Q

$P \rightarrow Q$ \rightarrow Intro 4-

R \rightarrow Elim 1,

$(P \rightarrow S) \rightarrow R$ \rightarrow Intro 3-

1. $(P \rightarrow Q) \rightarrow R$

2. $S \leftrightarrow Q$

3. $P \rightarrow S$ for \rightarrow Intro

4. P for \rightarrow Intro

5. S \rightarrow Elim 3,4

6. Q \leftrightarrow Elim 2,5

7. $P \rightarrow Q$ \rightarrow Intro 4-6

R \rightarrow Elim 1,

$(P \rightarrow S) \rightarrow R$ \rightarrow Intro 3-

1. $(P \rightarrow Q) \rightarrow R$

2. $S \leftrightarrow Q$

3. $P \rightarrow S$ for \rightarrow Intro

4. P for \rightarrow Intro

5. S \rightarrow Elim 3,4

6. Q \leftrightarrow Elim 2,5

7. $P \rightarrow Q$ \rightarrow Intro 4-6

8. R \rightarrow Elim 1,7

$(P \rightarrow S) \rightarrow R$ \rightarrow Intro 3-8

1. $(P \rightarrow Q) \rightarrow R$

2. $S \leftrightarrow Q$

3. $P \rightarrow S$ for \rightarrow Intro

4. P for \rightarrow Intro

5. S \rightarrow Elim 3,4

6. Q \leftrightarrow Elim 2,5

7. $P \rightarrow Q$ \rightarrow Intro 4-6

8. R \rightarrow Elim 1,7

9. $(P \rightarrow S) \rightarrow R$ \rightarrow Intro 3-8

EXAMPLE

Example:

$$P \leftrightarrow Q$$

$$Q \leftrightarrow R$$

$$(P \rightarrow R) \wedge (R \rightarrow P)$$

$$1. P \leftrightarrow Q$$

$$2. Q \leftrightarrow R$$

$$(P \rightarrow R) \wedge (R \rightarrow P)$$

EXAMPLE

Example:

$$\begin{array}{|l} P \leftrightarrow Q \\ Q \leftrightarrow R \\ \hline (P \rightarrow R) \wedge (R \rightarrow P) \end{array}$$

$$\begin{array}{|l} 1. P \leftrightarrow Q \\ 2. Q \leftrightarrow R \\ \hline \end{array}$$

3. P for \rightarrow Intro

$$(P \rightarrow R) \wedge (R \rightarrow P)$$

EXAMPLE

Example:

$$\begin{array}{|l} P \leftrightarrow Q \\ Q \leftrightarrow R \\ \hline (P \rightarrow R) \wedge (R \rightarrow P) \end{array}$$

$$1. P \leftrightarrow Q$$

$$2. Q \leftrightarrow R$$

$$\begin{array}{|l} 3. P \quad \text{for } \rightarrow \text{Intro} \\ \hline 4. Q \quad \leftrightarrow \text{Elim 1,3} \end{array}$$

$$(P \rightarrow R) \wedge (R \rightarrow P)$$

EXAMPLE

Example:

$$\begin{array}{|l} P \leftrightarrow Q \\ Q \leftrightarrow R \\ \hline (P \rightarrow R) \wedge (R \rightarrow P) \end{array}$$

$$1. P \leftrightarrow Q$$

$$2. Q \leftrightarrow R$$

$$\begin{array}{|l} 3. P \quad \text{for } \rightarrow \text{Intro} \\ \hline \end{array}$$

$$4. Q \quad \leftrightarrow \text{Elim } 1,3$$

$$5. R \quad \leftrightarrow \text{Elim } 2,4$$

$$(P \rightarrow R) \wedge (R \rightarrow P)$$

EXAMPLE

Example:

$$\begin{array}{|l} P \leftrightarrow Q \\ Q \leftrightarrow R \\ \hline (P \rightarrow R) \wedge (R \rightarrow P) \end{array}$$

$$1. P \leftrightarrow Q$$

$$2. Q \leftrightarrow R$$

$$\begin{array}{|l} 3. P \quad \text{for } \rightarrow \text{Intro} \\ \hline \end{array}$$

$$4. Q \quad \leftrightarrow \text{Elim } 1,3$$

$$5. R \quad \leftrightarrow \text{Elim } 2,4$$

$$6. P \rightarrow R \quad \rightarrow \text{Intro } 3-5$$

$$(P \rightarrow R) \wedge (R \rightarrow P)$$

EXAMPLE

Example:

$$\begin{array}{|l} P \leftrightarrow Q \\ Q \leftrightarrow R \\ \hline (P \rightarrow R) \wedge (R \rightarrow P) \end{array}$$

$$1. P \leftrightarrow Q$$

$$2. Q \leftrightarrow R$$

$$\begin{array}{|l} 3. P \quad \text{for } \rightarrow \text{Intro} \\ \hline \end{array}$$

$$4. Q \quad \leftrightarrow \text{Elim } 1,3$$

$$5. R \quad \leftrightarrow \text{Elim } 2,4$$

$$6. P \rightarrow R \quad \rightarrow \text{Intro } 3-5$$

$$\begin{array}{|l} 7. R \quad \text{for } \rightarrow \text{Intro} \\ \hline \end{array}$$

$$(P \rightarrow R) \wedge (R \rightarrow P)$$

EXAMPLE

Example:

$$\begin{array}{|l} P \leftrightarrow Q \\ Q \leftrightarrow R \\ \hline (P \rightarrow R) \wedge (R \rightarrow P) \end{array}$$

$$1. P \leftrightarrow Q$$

$$2. Q \leftrightarrow R$$

$$\begin{array}{|l} 3. P \quad \text{for } \rightarrow \text{Intro} \\ \hline \end{array}$$

$$4. Q \quad \leftrightarrow \text{Elim } 1,3$$

$$5. R \quad \leftrightarrow \text{Elim } 2,4$$

$$6. P \rightarrow R \quad \rightarrow \text{Intro } 3-5$$

$$\begin{array}{|l} 7. R \quad \text{for } \rightarrow \text{Intro} \\ \hline \end{array}$$

$$8. Q \quad \leftrightarrow \text{Elim } 2,7$$

$$(P \rightarrow R) \wedge (R \rightarrow P)$$

EXAMPLE

Example:

$$\begin{array}{|l} P \leftrightarrow Q \\ Q \leftrightarrow R \\ \hline (P \rightarrow R) \wedge (R \rightarrow P) \end{array}$$

$$1. P \leftrightarrow Q$$

$$2. Q \leftrightarrow R$$

$$\begin{array}{|l} 3. P \quad \text{for } \rightarrow \text{Intro} \\ \hline \end{array}$$

$$4. Q \quad \leftrightarrow \text{Elim } 1,3$$

$$5. R \quad \leftrightarrow \text{Elim } 2,4$$

$$6. P \rightarrow R \quad \rightarrow \text{Intro } 3-5$$

$$\begin{array}{|l} 7. R \quad \text{for } \rightarrow \text{Intro} \\ \hline \end{array}$$

$$8. Q \quad \leftrightarrow \text{Elim } 2,7$$

$$9. P \quad \leftrightarrow \text{Elim } 1,8$$

$$(P \rightarrow R) \wedge (R \rightarrow P)$$

EXAMPLE

Example:

$$\begin{array}{|l} P \leftrightarrow Q \\ Q \leftrightarrow R \\ \hline (P \rightarrow R) \wedge (R \rightarrow P) \end{array}$$

$$\begin{array}{|l} 1. P \leftrightarrow Q \\ 2. Q \leftrightarrow R \\ \hline \begin{array}{|l} 3. P \quad \text{for } \rightarrow \text{Intro} \\ \hline 4. Q \quad \leftrightarrow \text{Elim } 1,3 \\ 5. R \quad \leftrightarrow \text{Elim } 2,4 \\ 6. P \rightarrow R \quad \rightarrow \text{Intro } 3-5 \\ \hline 7. R \quad \text{for } \rightarrow \text{Intro} \\ \hline 8. Q \quad \leftrightarrow \text{Elim } 2,7 \\ 9. P \quad \leftrightarrow \text{Elim } 1,8 \\ 10. R \rightarrow P \quad \rightarrow \text{Intro } 7-9 \end{array} \\ \hline (P \rightarrow R) \wedge (R \rightarrow P) \end{array}$$

EXAMPLE

Example:

$$\begin{array}{|l} P \leftrightarrow Q \\ Q \leftrightarrow R \\ \hline (P \rightarrow R) \wedge (R \rightarrow P) \end{array}$$

$$1. P \leftrightarrow Q$$

$$2. Q \leftrightarrow R$$

$$3. P \quad \text{for } \rightarrow \text{Intro}$$

$$4. Q \quad \leftrightarrow \text{Elim } 1, 3$$

$$5. R \quad \leftrightarrow \text{Elim } 2, 4$$

$$6. P \rightarrow R \quad \rightarrow \text{Intro } 3-5$$

$$7. R \quad \text{for } \rightarrow \text{Intro}$$

$$8. Q \quad \leftrightarrow \text{Elim } 2, 7$$

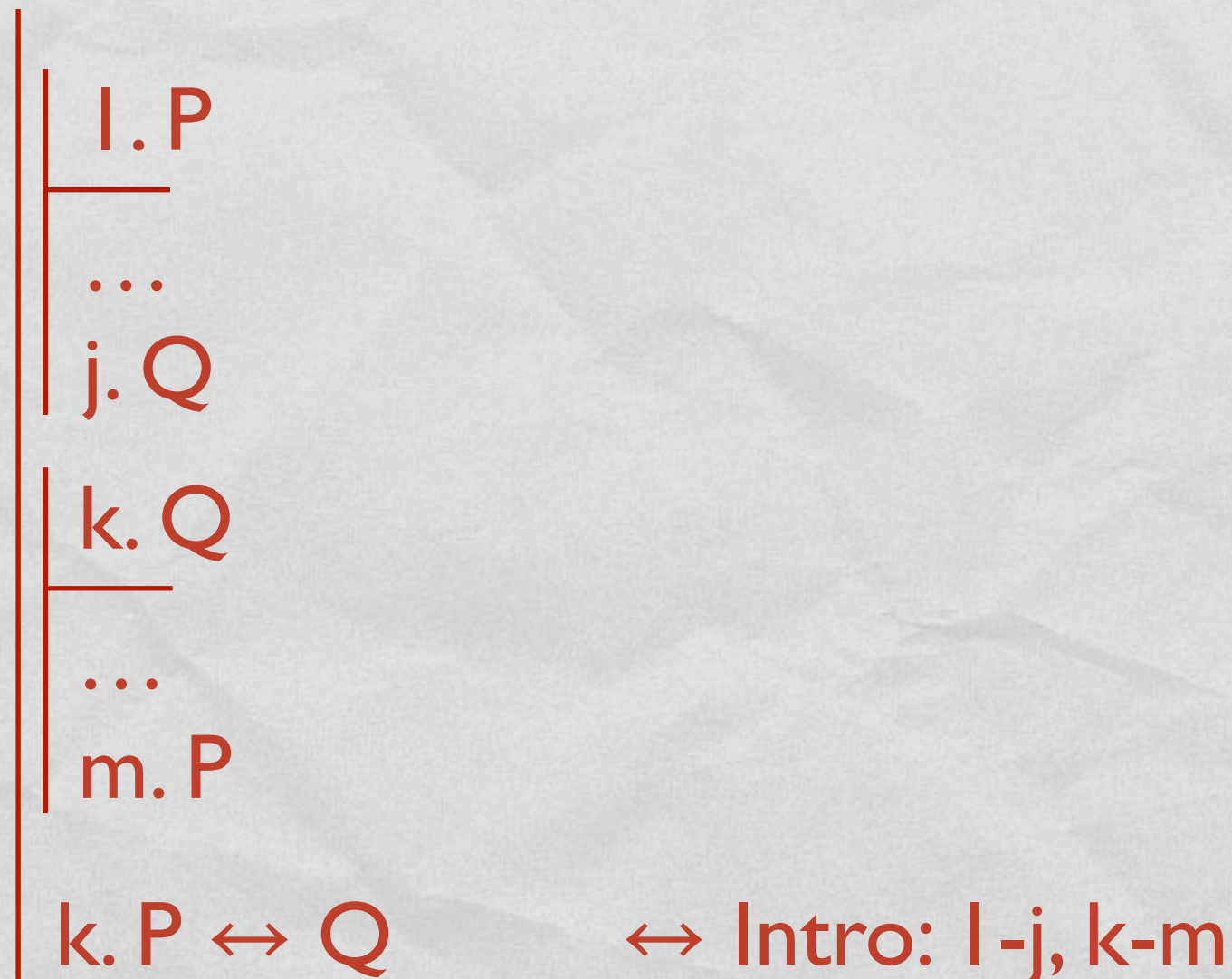
$$9. P \quad \leftrightarrow \text{Elim } 1, 8$$

$$10. R \rightarrow P \quad \rightarrow \text{Intro } 7-9$$

$$11. (P \rightarrow R) \wedge (R \rightarrow P) \quad \wedge \text{Intro } 6, 10$$

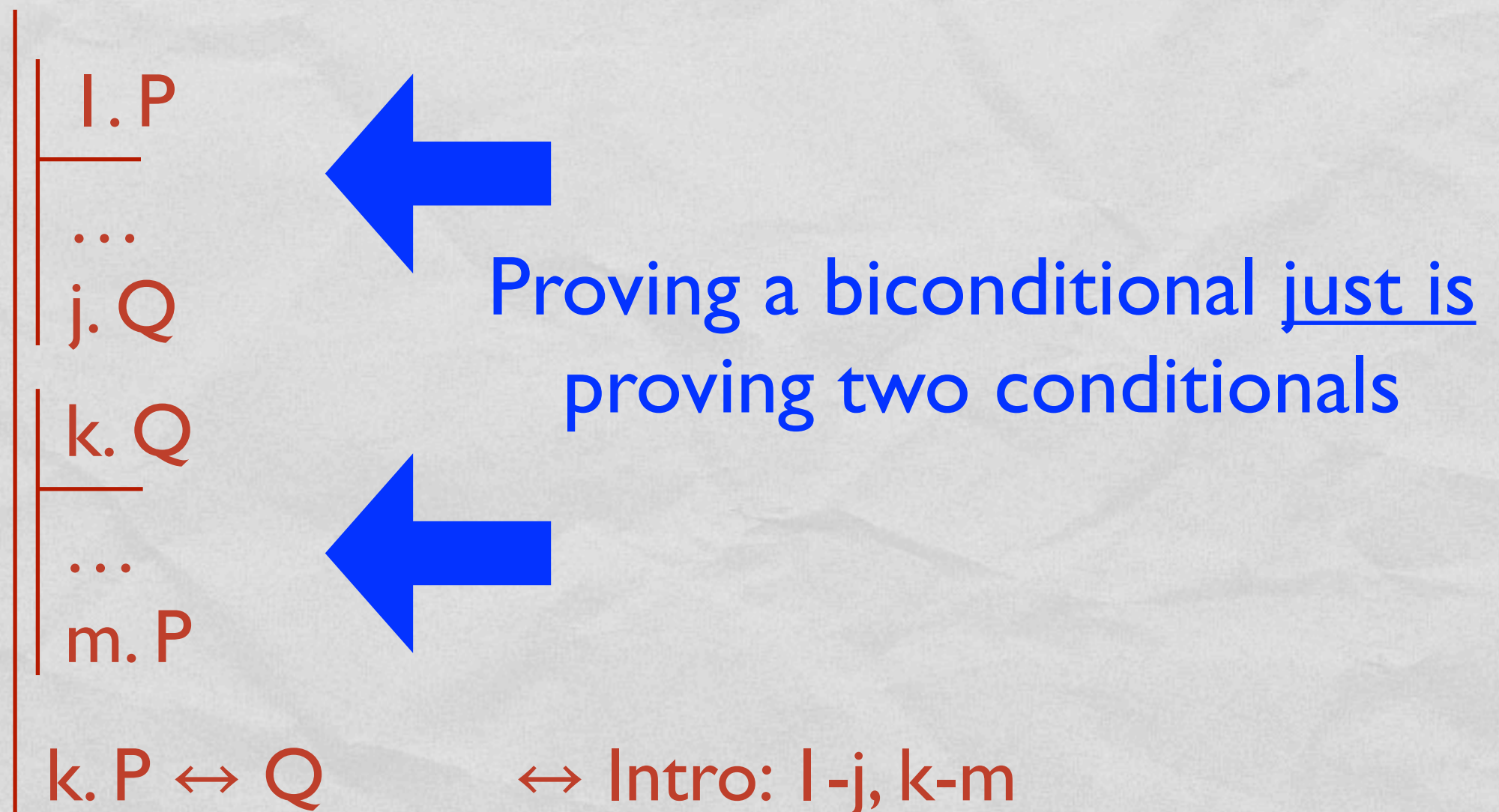
FORMAL PROOF RULES

- \leftrightarrow Introduction: from a proof from P to Q and a proof from Q to P , we can infer $P \leftrightarrow Q$.



FORMAL PROOF RULES

- \leftrightarrow Introduction: from a proof from P to Q and a proof from Q to P , we can infer $P \leftrightarrow Q$.



BICONDITIONAL INTRODUCTION

Example:

$$\begin{array}{|l} P \leftrightarrow Q \\ Q \leftrightarrow R \\ \hline P \leftrightarrow R \end{array}$$

1. $P \leftrightarrow Q$
2. $Q \leftrightarrow R$

$$P \leftrightarrow R$$

BICONDITIONAL INTRODUCTION

Example:

$$\begin{array}{|l} P \leftrightarrow Q \\ Q \leftrightarrow R \\ \hline P \leftrightarrow R \end{array}$$

$$\begin{array}{|l} 1. P \leftrightarrow Q \\ 2. Q \leftrightarrow R \\ \hline \end{array}$$

3. P for \leftrightarrow Intro

$$P \leftrightarrow R$$

BICONDITIONAL INTRODUCTION

Example:

$$\begin{array}{|l} P \leftrightarrow Q \\ Q \leftrightarrow R \\ \hline P \leftrightarrow R \end{array}$$

1. $P \leftrightarrow Q$

2. $Q \leftrightarrow R$

3. P for \leftrightarrow Intro

4. Q \leftrightarrow Elim 1,3

$P \leftrightarrow R$

BICONDITIONAL INTRODUCTION

Example:

$$\begin{array}{|l} P \leftrightarrow Q \\ Q \leftrightarrow R \\ \hline P \leftrightarrow R \end{array}$$

1. $P \leftrightarrow Q$

2. $Q \leftrightarrow R$

3. P for \leftrightarrow Intro

4. Q \leftrightarrow Elim 1,3

5. R \leftrightarrow Elim 2,4

$P \leftrightarrow R$

BICONDITIONAL INTRODUCTION

Example:

$$\begin{array}{|l} P \leftrightarrow Q \\ Q \leftrightarrow R \\ \hline P \leftrightarrow R \end{array}$$
$$\begin{array}{|l} 1. P \leftrightarrow Q \\ 2. Q \leftrightarrow R \\ \hline \end{array}$$
$$\begin{array}{|l} 3. P \quad \text{for } \leftrightarrow \text{Intro} \\ \hline \end{array}$$
$$\begin{array}{|l} 4. Q \quad \leftrightarrow \text{Elim } 1,3 \\ \hline \end{array}$$
$$\begin{array}{|l} 5. R \quad \leftrightarrow \text{Elim } 2,4 \\ \hline \end{array}$$
$$\begin{array}{|l} 6. R \quad \text{for } \leftrightarrow \text{Intro} \\ \hline \end{array}$$
$$P \leftrightarrow R$$

BICONDITIONAL INTRODUCTION

Example:

$$\begin{array}{|l} P \leftrightarrow Q \\ Q \leftrightarrow R \\ \hline P \leftrightarrow R \end{array}$$

$$1. P \leftrightarrow Q$$

$$2. Q \leftrightarrow R$$

$$\begin{array}{|l} 3. P \quad \text{for } \leftrightarrow \text{Intro} \\ \hline \end{array}$$

$$4. Q \quad \leftrightarrow \text{Elim } 1,3$$

$$5. R \quad \leftrightarrow \text{Elim } 2,4$$

$$\begin{array}{|l} 6. R \quad \text{for } \leftrightarrow \text{Intro} \\ \hline \end{array}$$

$$7. Q \quad \leftrightarrow \text{Elim } 2,6$$

$$P \leftrightarrow R$$

BICONDITIONAL INTRODUCTION

Example:

$P \leftrightarrow Q$
$Q \leftrightarrow R$
<hr/>
$P \leftrightarrow R$

1. $P \leftrightarrow Q$

2. $Q \leftrightarrow R$

3. P	for \leftrightarrow Intro
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4. Q	\leftrightarrow Elim 1,3
--------	----------------------------

5. R	\leftrightarrow Elim 2,4
--------	----------------------------

6. R	for \leftrightarrow Intro
--------	-----------------------------

7. Q	\leftrightarrow Elim 2,6
--------	----------------------------

8. P	\leftrightarrow Elim 1,7
--------	----------------------------

$P \leftrightarrow R$

BICONDITIONAL INTRODUCTION

Example:

$P \leftrightarrow Q$
$Q \leftrightarrow R$
<hr/>
$P \leftrightarrow R$

1. $P \leftrightarrow Q$

2. $Q \leftrightarrow R$

3. P for \leftrightarrow Intro

4. Q \leftrightarrow Elim 1,3

5. R \leftrightarrow Elim 2,4

6. R for \leftrightarrow Intro

7. Q \leftrightarrow Elim 2,6

8. P \leftrightarrow Elim 1,7

9. $P \leftrightarrow R$ \leftrightarrow Intro 3-5, 6-8

1. $P \leftrightarrow Q$

2. $Q \leftrightarrow R$

3. P for \rightarrow Intro

4. Q \leftrightarrow Elim 1,3

5. R \leftrightarrow Elim 2,4

6. $P \rightarrow R$ \rightarrow Intro 3-5

7. R for \rightarrow Intro

8. Q \leftrightarrow Elim 2,7

9. P \leftrightarrow Elim 1,8

10. $R \rightarrow P$ \rightarrow Intro 7-9

11. $(P \rightarrow R) \wedge (R \rightarrow P)$ \wedge Intro 6,10

1. $P \leftrightarrow Q$

2. $Q \leftrightarrow R$

3. P for \leftrightarrow Intro

4. Q \leftrightarrow Elim 1,3

5. R \leftrightarrow Elim 2,4

6. R for \leftrightarrow Intro

7. Q \leftrightarrow Elim 2,6

8. P \leftrightarrow Elim 1,7

9. $P \leftrightarrow R$ \leftrightarrow Intro 3-5, 6-8

BICONDITIONAL INTRODUCTION

Example:

$$\begin{array}{|l} P \leftrightarrow Q \\ \hline \neg P \leftrightarrow \neg Q \end{array}$$

BICONDITIONAL INTRODUCTION

Example:

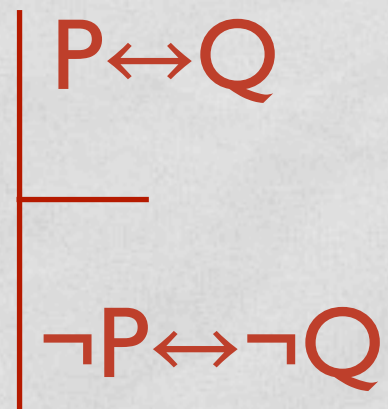
$P \leftrightarrow Q$
—
 $\neg P \leftrightarrow \neg Q$

1. $P \leftrightarrow Q$
—

$\neg P \leftrightarrow \neg Q$

BICONDITIONAL INTRODUCTION

Example:



1. $P \leftrightarrow Q$

2. $\neg P$

for \leftrightarrow Intro

$\neg Q$

$\neg Q$

for \leftrightarrow Intro

$\neg P$

$\neg P \leftrightarrow \neg Q$

\leftrightarrow Intro

$$1. P \leftrightarrow Q$$

$$\neg P \leftrightarrow \neg Q$$

1. $P \leftrightarrow Q$

2. $\neg P$

for \leftrightarrow Intro

$\neg Q$

$\neg P \leftrightarrow \neg Q$

\leftrightarrow Intro

1. $P \leftrightarrow Q$

2. $\neg P$

for \leftrightarrow Intro

3. Q

for \neg Intro

$\neg Q$

\neg Intro

$\neg P \leftrightarrow \neg Q$

\leftrightarrow Intro

1. $P \leftrightarrow Q$

2. $\neg P$ for \leftrightarrow Intro

3. Q for \neg Intro

4. P \leftrightarrow Elim 1,3

$\neg Q$ \neg Intro

$\neg P \leftrightarrow \neg Q$ \leftrightarrow Intro

1. $P \leftrightarrow Q$

2. $\neg P$ for \leftrightarrow Intro

3. Q for \neg Intro

4. P \leftrightarrow Elim 1,3

5. \perp \perp Intro 2,4

$\neg Q$ \neg Intro

$\neg P \leftrightarrow \neg Q$ \leftrightarrow Intro

1. $P \leftrightarrow Q$

2. $\neg P$ for \leftrightarrow Intro

3. Q for \neg Intro

4. P \leftrightarrow Elim 1,3

5. \perp \perp Intro 2,4

6. $\neg Q$ \neg Intro 3-5

$\neg P \leftrightarrow \neg Q$ \leftrightarrow Intro

1. $P \leftrightarrow Q$

2. $\neg P$ for \leftrightarrow Intro

3. Q for \neg Intro

4. P \leftrightarrow Elim 1,3

5. \perp \perp Intro 2,4

6. $\neg Q$ \neg Intro 3-5

7. $\neg Q$ for \leftrightarrow Intro

$\neg P$

$\neg P \leftrightarrow \neg Q$ \leftrightarrow Intro

1. $P \leftrightarrow Q$

2. $\neg P$ for \leftrightarrow Intro

3. Q for \neg Intro

4. P \leftrightarrow Elim 1,3

5. \perp \perp Intro 2,4

6. $\neg Q$ \neg Intro 3-5

7. $\neg Q$ for \leftrightarrow Intro

8. P for \neg Intro

$\neg P$

$\neg P \leftrightarrow \neg Q$ \leftrightarrow Intro

1. $P \leftrightarrow Q$

2. $\neg P$ for \leftrightarrow Intro

3. Q for \neg Intro

4. P \leftrightarrow Elim 1,3

5. \perp \perp Intro 2,4

6. $\neg Q$ \neg Intro 3-5

7. $\neg Q$ for \leftrightarrow Intro

8. P for \neg Intro

9. Q \leftrightarrow Elim 1,8

$\neg P$

$\neg P \leftrightarrow \neg Q$ \leftrightarrow Intro

1. $P \leftrightarrow Q$

2. $\neg P$ for \leftrightarrow Intro

3. Q for \neg Intro

4. P \leftrightarrow Elim 1,3

5. \perp \perp Intro 2,4

6. $\neg Q$ \neg Intro 3-5

7. $\neg Q$ for \leftrightarrow Intro

8. P for \neg Intro

9. Q \leftrightarrow Elim 1,8

10. \perp \perp Intro 7,9

$\neg P$

$\neg P \leftrightarrow \neg Q$ \leftrightarrow Intro

1. $P \leftrightarrow Q$

2. $\neg P$ for \leftrightarrow Intro

3. Q for \neg Intro

4. P \leftrightarrow Elim 1,3

5. \perp \perp Intro 2,4

6. $\neg Q$ \neg Intro 3-5

7. $\neg Q$ for \leftrightarrow Intro

8. P for \neg Intro

9. Q \leftrightarrow Elim 1,8

10. \perp \perp Intro 7,9

11. $\neg P$ \neg Intro 8-10

$\neg P \leftrightarrow \neg Q$ \leftrightarrow Intro

1. $P \leftrightarrow Q$

2. $\neg P$ for \leftrightarrow Intro

3. Q for \neg Intro

4. P \leftrightarrow Elim 1,3

5. \perp \perp Intro 2,4

6. $\neg Q$ \neg Intro 3-5

7. $\neg Q$ for \leftrightarrow Intro

8. P for \neg Intro

9. Q \leftrightarrow Elim 1,8

10. \perp \perp Intro 7,9

11. $\neg P$ \neg Intro 8-10

12. $\neg P \leftrightarrow \neg Q$ \leftrightarrow Intro 2-6, 7-11

BICONDITIONAL INTRODUCTION

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- When you prove a biconditional, you are showing that you can do two proofs - one from left to right and one from right to left.

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- When you prove a biconditional, you are showing that you can do two proofs - one from left to right and one from right to left.
- If two sentences are equivalent, then you could do a proof from the first to the second and you could also do a proof from the second to the first.

TABLES AND PROOFS

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- If a sentence is a tautology, you can prove it from no premises at all.

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- If two sentences are equivalent, then the biconditional between them is a tautology

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- If a sentence is a tautology, you can prove it from no premises at all.
- If two sentences are equivalent, then the biconditional between them is a tautology
 - ---- So you can prove the biconditional from no premises at all (by doing the two relevant proofs and then sticking them together)

EXAMPLES OF EQUIVALENCES

DeMorgan's Laws

$$\begin{array}{|l} \neg(P \vee Q) \\ \hline \neg P \wedge \neg Q \end{array}$$

and also

$$\begin{array}{|l} \neg P \wedge \neg Q \\ \hline \neg(P \vee Q) \end{array}$$

so by doing both proofs and then doing \leftrightarrow Intro

$$\begin{array}{|l} \neg(P \vee Q) \leftrightarrow (\neg P \wedge \neg Q) \end{array}$$

EXAMPLES OF EQUIVALENCES

Contraposition

$$\begin{array}{|l} P \rightarrow Q \\ \hline \neg Q \rightarrow \neg P \end{array}$$

and also

$$\begin{array}{|l} \neg Q \rightarrow \neg P \\ \hline P \rightarrow Q \end{array}$$

so by doing both proofs and then doing \leftrightarrow Intro

$$\begin{array}{|l} \hline (P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P) \end{array}$$