You are given four cards with a number on one side and a letter on the other. You can only see one side of each card.

Which card(s) do you have to turn over in order to fully test the following rule:

If there is a vowel on one side of the card,
then there is an even number on the other side.

You are given four cards with a drink on one side and an age on the other. You can only see one side of each card.

Which card(s) do you have to turn over in order to fully test the following rule:

If you are drinking alcohol, then you must be over 21.

Rules for Conditionals

1. $P \rightarrow Q$	1. $P \rightarrow Q$
2. P	2. $\neg Q$
3. Q $\rightarrow$ Elim: 1,2	3. $\neg P$ VALID, but not $\rightarrow$E

Modus Ponens

Modus Tollens

Proofs with Conditionals II

Wednesday, 26 February

Rules for Conditionals

- → Elimination: from P → Q and P, we can infer Q.

- ↔ Elimination: from P ↔ Q and P/Q, we can infer Q/P.

Rules for Conditionals

- → Elimination: from P → Q and P, we can infer Q.

1. P → Q	
2. P	
3. Q	→ Elim: 1,2

- ↔ Elimination: from P ↔ Q and P/Q, we can infer Q/P.

Rules for Conditionals

- → Elimination: from P → Q and P, we can infer Q.

1. P → Q	
2. P	
3. Q	→ Elim: 1,2

- ↔ Elimination: from P ↔ Q and P/Q, we can infer Q/P.

Rules for Conditionals

- → Elimination: from P → Q and P, we can infer Q.

1. P → Q	
2. P	
3. Q	→ Elim: 1,2

- ↔ Elimination: from P ↔ Q and P/Q, we can infer Q/P.

1. P ↔ Q	
2. Q	
3. P	↔ Elim: 1,2

Example:

$D \rightarrow A$

$D \leftrightarrow B$

$C \leftrightarrow (B \wedge \neg A)$

$\neg C$

Example:

$D \rightarrow A$

$D \leftrightarrow B$

$C \leftrightarrow (B \wedge \neg A)$

$\neg C$

1. $D \rightarrow A$
2. $D \leftrightarrow B$
3. $C \leftrightarrow (B \wedge \neg A)$

$\neg C$

Example:

D → A
D ↔ B
C ↔ (B ∧ ¬A)

¬C

1. D → A
2. D ↔ B
3. C ↔ (B ∧ ¬A)

---

4. C

for ¬Intro

¬C

¬Intro

Example:

D → A
D ↔ B
C ↔ (B ∧ ¬A)

¬C

1. D → A
2. D ↔ B
3. C ↔ (B ∧ ¬A)

---

4. C — for ¬Intro

---

5. B ∧ ¬A — ↔ Elim 3,4

¬C — ¬Intro

Example:

D → A
D ↔ B
C ↔ (B ∧ ¬A)

¬C

1. D → A
2. D ↔ B
3. C ↔ (B ∧ ¬A)

4. C — for ¬Intro
5. B ∧ ¬A — ↔ Elim 3,4
6. B — ∧ Elim 5

¬C — ¬Intro

Example:

D → A
D ↔ B
C ↔ (B ∧ ¬A)

¬C

1. D → A
2. D ↔ B
3. C ↔ (B ∧ ¬A)

---

4. C — for ¬Intro

---

5. B ∧ ¬A — ↔ Elim 3,4
6. B — ∧ Elim 5
7. ¬A — ∧ Elim 5

¬C — ¬Intro

Example:

D → A
D ↔ B
C ↔ (B ∧ ¬A)

¬C

1. D → A
2. D ↔ B
3. C ↔ (B ∧ ¬A)

---

4. C — for ¬Intro

---

5. B ∧ ¬A — ↔ Elim 3,4
6. B — ∧ Elim 5
7. ¬A — ∧ Elim 5
8. D — ↔ Elim 2,6

¬C — ¬Intro

Example:

D → A
D ↔ B
C ↔ (B ∧ ¬A)

¬C

1. D → A
2. D ↔ B
3. C ↔ (B ∧ ¬A)

---

4. C — for ¬Intro

---

5. B ∧ ¬A — ↔ Elim 3,4
6. B — ∧ Elim 5
7. ¬A — ∧ Elim 5
8. D — ↔ Elim 2,6
9. A — → Elim 1,8

¬C — ¬Intro

Example:

D → A
D ↔ B
C ↔ (B ∧ ¬A)

¬C

1. D → A
2. D ↔ B
3. C ↔ (B ∧ ¬A)

---

4. C — for ¬Intro

---

5. B ∧ ¬A — ↔ Elim 3,4
6. B — ∧ Elim 5
7. ¬A — ∧ Elim 5
8. D — ↔ Elim 2,6
9. A — → Elim 1,8
10. ⊥ — ⊥ Intro 7,9

¬C — ¬Intro

Example:

D → A
D ↔ B
C ↔ (B ∧ ¬A)

¬C

1. D → A
2. D ↔ B
3. C ↔ (B ∧ ¬A)

---

4. C — for ¬Intro

---

5. B ∧ ¬A — ↔ Elim 3,4
6. B — ∧ Elim 5
7. ¬A — ∧ Elim 5
8. D — ↔ Elim 2,6
9. A — → Elim 1,8
10. ⊥ — ⊥ Intro 7,9

11. ¬C — ¬ Intro 4-10

Conditional Introduction

Conditional Introduction

- → Introduction

  From a proof from P to Q, we can infer P → Q.

Conditional Introduction

- → Introduction

  From a proof from P to Q, we can infer P → Q.

  | l. P
  |
  | ...
  | j. Q
  |
  k. P → Q → Intro: l-j

Conditional Introduction

- → Introduction

  From a proof from P to Q, we can infer P → Q.

  | l. P
  |
  | …
  | j. Q
  |
  | k. P → Q → Intro: l-j

This rule is often known as Conditional Proof

Example

Example:

$$\frac{P \rightarrow (Q \leftrightarrow R) \quad R}{P \rightarrow Q}$$

Example

Example:

$P \rightarrow (Q \leftrightarrow R)$

R

$P \rightarrow Q$

1. $P \rightarrow (Q \leftrightarrow R)$
2. R

$P \rightarrow Q$

EXAMPLE

Example:

$P \rightarrow (Q \leftrightarrow R)$

R

$P \rightarrow Q$

1. $P \rightarrow (Q \leftrightarrow R)$
2. R
3. P
4. Q
5. $P \rightarrow Q$ for $\rightarrow$Intro

$\rightarrow$Intro

EXAMPLE

Example:

$P \rightarrow (Q \leftrightarrow R)$

R

$P \rightarrow Q$

1. $P \rightarrow (Q \leftrightarrow R)$
2. R
3. P for $\rightarrow$Intro
4. $Q \leftrightarrow R$ $\rightarrow$ Elim 1,3
5. Q
6. $P \rightarrow Q$ $\rightarrow$Intro

EXAMPLE

Example:

$P \rightarrow (Q \leftrightarrow R)$

R

$P \rightarrow Q$

1. $P \rightarrow (Q \leftrightarrow R)$
2. R
3. P for $\rightarrow$Intro
4. $Q \leftrightarrow R$ $\rightarrow$ Elim 1,3
5. Q $\leftrightarrow$ Elim 2,4

$P \rightarrow Q$ $\rightarrow$Intro

EXAMPLE

Example:

$P \rightarrow (Q \leftrightarrow R)$

R

$P \rightarrow Q$

1. $P \rightarrow (Q \leftrightarrow R)$
2. R

3. P — for →Intro

4. $Q \leftrightarrow R$ — → Elim 1,3

5. Q — ↔ Elim 2,4

6. $P \rightarrow Q$ — →Intro 3-5

Conditionals with Conjunctions

Example:

$(P \rightarrow Q) \wedge (P \rightarrow R)$

$P \rightarrow (Q \wedge R)$

Conditionals with Conjunctions

Example:

$(P \rightarrow Q) \wedge (P \rightarrow R)$

$P \rightarrow (Q \wedge R)$

1. $(P \rightarrow Q) \wedge (P \rightarrow R)$

$P \rightarrow (Q \wedge R)$

Conditionals with Conjunctions

Example:

$(P \rightarrow Q) \wedge (P \rightarrow R)$

$P \rightarrow (Q \wedge R)$

1. $(P \rightarrow Q) \wedge (P \rightarrow R)$
2. $P \rightarrow Q$ $\wedge$ Elim 1

$P \rightarrow (Q \wedge R)$

Conditionals with Conjunctions

Example:

$(P \rightarrow Q) \wedge (P \rightarrow R)$

$P \rightarrow (Q \wedge R)$

1. $(P \rightarrow Q) \wedge (P \rightarrow R)$

---

2. $P \rightarrow Q$ $\wedge$ Elim 1
3. $P \rightarrow R$ $\wedge$ Elim 1

$P \rightarrow (Q \wedge R)$

Conditionals with Conjunctions

Example:

$(P \rightarrow Q) \wedge (P \rightarrow R)$

$P \rightarrow (Q \wedge R)$

1. $(P \rightarrow Q) \wedge (P \rightarrow R)$

---

2. $P \rightarrow Q$ $\wedge$ Elim 1
3. $P \rightarrow R$ $\wedge$ Elim 1
4. P for $\rightarrow$Intro

$Q \wedge R$

$P \rightarrow (Q \wedge R)$ $\rightarrow$Intro

Conditionals with Conjunctions

Example:

$(P \rightarrow Q) \wedge (P \rightarrow R)$

$P \rightarrow (Q \wedge R)$

1. $(P \rightarrow Q) \wedge (P \rightarrow R)$

---

2. $P \rightarrow Q$ $\wedge$ Elim 1
3. $P \rightarrow R$ $\wedge$ Elim 1
4. P for $\rightarrow$Intro

---

5. Q $\rightarrow$ Elim 2,4

$Q \wedge R$

$P \rightarrow (Q \wedge R)$ $\rightarrow$Intro

Conditionals with Conjunctions

Example:

$(P \rightarrow Q) \wedge (P \rightarrow R)$

$P \rightarrow (Q \wedge R)$

1. $(P \rightarrow Q) \wedge (P \rightarrow R)$

---

2. $P \rightarrow Q$ $\wedge$ Elim 1
3. $P \rightarrow R$ $\wedge$ Elim 1
4. P for $\rightarrow$Intro

---

5. Q $\rightarrow$ Elim 2,4
6. R $\rightarrow$ Elim 3,4

$Q \wedge R$

$P \rightarrow (Q \wedge R)$ $\rightarrow$Intro

Conditionals with Conjunctions

Example:

$(P \rightarrow Q) \wedge (P \rightarrow R)$

$P \rightarrow (Q \wedge R)$

1. $(P \rightarrow Q) \wedge (P \rightarrow R)$

---

2. $P \rightarrow Q$ $\wedge$ Elim 1
3. $P \rightarrow R$ $\wedge$ Elim 1
4. P for $\rightarrow$Intro

---

5. Q $\rightarrow$ Elim 2,4
6. R $\rightarrow$ Elim 3,4
7. $Q \wedge R$ $\wedge$ Intro 5,6

$P \rightarrow (Q \wedge R)$ $\rightarrow$Intro

Conditionals with Conjunctions

Example:

$(P \rightarrow Q) \wedge (P \rightarrow R)$

$P \rightarrow (Q \wedge R)$

1.	$(P \rightarrow Q) \wedge (P \rightarrow R)$	
2.	$P \rightarrow Q$	$\wedge$ Elim 1
3.	$P \rightarrow R$	$\wedge$ Elim 1
4.	P	for $\rightarrow$Intro
5.	Q	$\rightarrow$ Elim 2,4
6.	R	$\rightarrow$ Elim 3,4
7.	$Q \wedge R$	$\wedge$ Intro 5,6
8.	$P \rightarrow (Q \wedge R)$	$\rightarrow$ Elim 4-7

Conditionals with Conjunctions

Example:

$(P \rightarrow Q) \wedge (P \rightarrow R)$

$P \rightarrow (Q \wedge R)$

In fact these are equivalent

1. $(P \rightarrow Q) \wedge (P \rightarrow R)$
2. $P \rightarrow Q$ $\wedge$ Elim 1
3. $P \rightarrow R$ $\wedge$ Elim 1
4. P for $\rightarrow$Intro
5. Q $\rightarrow$ Elim 2,4
6. R $\rightarrow$ Elim 3,4
7. $Q \wedge R$ $\wedge$ Intro 5,6
8. $P \rightarrow (Q \wedge R)$ $\rightarrow$ Elim 4-7

Conditionals with Disjunctions

Example:

$(P \rightarrow Q) \vee (P \rightarrow R)$

$P \rightarrow (Q \vee R)$

Conditionals with Disjunctions

Example:

$(P \rightarrow Q) \vee (P \rightarrow R)$

$P \rightarrow (Q \vee R)$

1. $(P \rightarrow Q) \vee (P \rightarrow R)$

$P \rightarrow (Q \vee R)$

(P → Q) ∨ (P → R)

P → (Q ∨ R)

1. (P → Q) ∨ (P → R)

P → (Q ∨ R)

(P → Q) ∨ (P → R)

P → (Q ∨ R)

1. (P → Q) ∨ (P → R)

2. P

for →Intro

Q ∨ R

P → (Q ∨ R) →Intro

(P → Q) ∨ (P → R)

P → (Q ∨ R)

1. (P → Q) ∨ (P → R)

2. P for →Intro

3. P → Q for ∨ Elim

Q ∨ R

P → R for ∨ Elim

Q ∨ R

Q ∨ R ∨ Elim

P → (Q ∨ R) →Intro

(P → Q) ∨ (P → R)

P → (Q ∨ R)

1. (P → Q) ∨ (P → R)

2. P for →Intro

3. P → Q for ∨ Elim

4. Q → Elim 2,3

Q ∨ R

P → R for ∨ Elim

Q ∨ R

Q ∨ R ∨ Elim

P → (Q ∨ R) →Intro

(P → Q) ∨ (P → R)

P → (Q ∨ R)

1. (P → Q) ∨ (P → R)

2. P for →Intro

3. P → Q for ∨ Elim

4. Q → Elim 2,3

5. Q ∨ R ∨ Intro 4

P → R for ∨ Elim

Q ∨ R

Q ∨ R ∨ Elim

P → (Q ∨ R) →Intro

(P → Q) ∨ (P → R)

P → (Q ∨ R)

1. (P → Q) ∨ (P → R)

2. P for →Intro

3. P → Q for ∨ Elim

4. Q → Elim 2,3

5. Q ∨ R ∨ Intro 4

6. P → R for ∨ Elim

7. R → Elim 2,6

Q ∨ R

Q ∨ R ∨ Elim

P → (Q ∨ R) →Intro

(P → Q) ∨ (P → R)

P → (Q ∨ R)

1. (P → Q) ∨ (P → R)

2. P for →Intro

3. P → Q for ∨ Elim

4. Q → Elim 2,3

5. Q ∨ R ∨ Intro 4

6. P → R for ∨ Elim

7. R → Elim 2,6

8. Q ∨ R ∨ Intro 7

Q ∨ R ∨ Elim

P → (Q ∨ R) →Intro

(P → Q) ∨ (P → R)

P → (Q ∨ R)

1. (P → Q) ∨ (P → R)

2. P for →Intro

3. P → Q for ∨ Elim

4. Q → Elim 2,3

5. Q ∨ R ∨ Intro 4

6. P → R for ∨ Elim

7. R → Elim 2,6

8. Q ∨ R ∨ Intro 7

9. Q ∨ R ∨ Elim 1, 3-5, 6-8

P → (Q ∨ R) →Intro

(P → Q) ∨ (P → R)

P → (Q ∨ R)

1. (P → Q) ∨ (P → R)

2. P for →Intro

3. P → Q for ∨ Elim

4. Q → Elim 2,3

5. Q ∨ R ∨ Intro 4

6. P → R for ∨ Elim

7. R → Elim 2,6

8. Q ∨ R ∨ Intro 7

9. Q ∨ R ∨ Elim 1, 3-5, 6-8

10. P → (Q ∨ R) → Intro 2-9

(P → Q) ∨ (P → R)

P → (Q ∨ R)

In fact these are equivalent

1. (P → Q) ∨ (P → R)
2. P — for →Intro
3. P → Q — for ∨ Elim
4. Q — → Elim 2,3
5. Q ∨ R — ∨ Intro 4
6. P → R — for ∨ Elim
7. R — → Elim 2,6
8. Q ∨ R — ∨ Intro 7
9. Q ∨ R — ∨ Elim 1, 3-5, 6-8
10. P → (Q ∨ R) — → Intro 2-9

Contraposition

Example:

$$\frac{P \rightarrow Q}{\neg Q \rightarrow \neg P}$$

Contraposition

Example:

$P \rightarrow Q$

$\neg Q \rightarrow \neg P$

1. $P \rightarrow Q$

$\neg Q \rightarrow \neg P$

Contraposition

Example:

$P \rightarrow Q$

$\neg Q \rightarrow \neg P$

1. $P \rightarrow Q$

2. $\neg Q$

6. $\neg P$

$\neg Q \rightarrow \neg P$ $\rightarrow$Intro

for $\rightarrow$Intro

Contraposition

Example:

$P \rightarrow Q$

$\neg Q \rightarrow \neg P$

1. $P \rightarrow Q$
2. $\neg Q$ — for $\rightarrow$Intro
3. P — for $\neg$Intro
6. $\neg P$ — $\neg$Intro

$\neg Q \rightarrow \neg P$ — $\rightarrow$Intro

Contraposition

Example:

$P \rightarrow Q$

$\neg Q \rightarrow \neg P$

1. $P \rightarrow Q$	
2. $\neg Q$	for $\rightarrow$Intro
3. P	for $\neg$Intro
4. Q	$\rightarrow$ Elim 1,3
6. $\neg P$	$\neg$Intro
$\neg Q \rightarrow \neg P$	$\rightarrow$Intro

Contraposition

Example:

$P \rightarrow Q$

$\neg Q \rightarrow \neg P$

1. $P \rightarrow Q$	
2. $\neg Q$	for $\rightarrow$Intro
3. P	for $\neg$Intro
4. Q	$\rightarrow$ Elim 1,3
5. $\perp$	$\perp$Intro 2,4
6. $\neg P$	$\neg$Intro
$\neg Q \rightarrow \neg P$	$\rightarrow$Intro

Contraposition

Example:

$P \rightarrow Q$

$\neg Q \rightarrow \neg P$

1. $P \rightarrow Q$	
2. $\neg Q$	for $\rightarrow$Intro
3. P	for $\neg$Intro
4. Q	$\rightarrow$ Elim 1,3
5. $\bot$	$\bot$Intro 2,4
6. $\neg P$	$\neg$ Intro 3-5
7. $\neg Q \rightarrow \neg P$	$\rightarrow$Intro 2-6

Modus Tollens

Example:

$P \rightarrow Q$

$\neg Q$

$\neg P$

1. $P \rightarrow Q$
2. $\neg Q$

---

3. P

---

4. Q → Elim 1,3
5. $\bot$ $\bot$Intro 2,4

6. $\neg P$ ¬ Intro 3-5

for ¬Intro

Modus Tollens vs. Contraposition

1. $P \rightarrow Q$	
2. $\neg Q$	
3. P	for ¬Intro
4. Q	→ Elim 1,3
5. $\bot$	⊥Intro 2,4
6. $\neg P$	¬ Intro 3-5

1. $P \rightarrow Q$	
2. $\neg Q$	for →Intro
3. P	for ¬Intro
4. Q	→ Elim 1,3
5. $\bot$	⊥Intro 2,4
6. $\neg P$	¬ Intro 3-5
7. $\neg Q \rightarrow \neg P$	→Intro 2-6

Modus Tollens vs. Contraposition

Modus Tollens

$$\begin{array}{l} P \rightarrow Q \\ \neg Q \\ \hline \neg P \end{array}$$

Contraposition

$$\begin{array}{l} P \rightarrow Q \\ \hline \neg Q \rightarrow \neg P \end{array}$$

A conditional proof transforms one problem into a slightly shorter problem with one extra premise

Example

Example:

$P \rightarrow Q$

$Q \rightarrow R$

P

R

Example

Example:

$P \rightarrow Q$

$Q \rightarrow R$

P

R

1. $P \rightarrow Q$
2. $Q \rightarrow R$
3. P

Example

Example:

$P \rightarrow Q$

$Q \rightarrow R$

P

R

1. $P \rightarrow Q$
2. $Q \rightarrow R$
3. P

---

4. Q $\rightarrow$ Elim 1,3

Example

Example:

$$\frac{P \rightarrow Q \quad Q \rightarrow R \quad P}{R}$$

1. $P \rightarrow Q$
2. $Q \rightarrow R$
3. P

---

4. Q $\rightarrow$ Elim 1,3
5. R $\rightarrow$ Elim 2,4

Chain Argument

Example:

$$\frac{P \rightarrow Q \qquad Q \rightarrow R}{P \rightarrow R}$$

Chain Argument

Example:

$P \rightarrow Q$

$Q \rightarrow R$

$P \rightarrow R$

1. $P \rightarrow Q$
2. $Q \rightarrow R$

Chain Argument

Example:

$P \rightarrow Q$

$Q \rightarrow R$

$P \rightarrow R$

1. $P \rightarrow Q$
2. $Q \rightarrow R$

---

3. P

for $\rightarrow$Intro

Chain Argument

Example:

$P \rightarrow Q$

$Q \rightarrow R$

$P \rightarrow R$

1. $P \rightarrow Q$
2. $Q \rightarrow R$

---

3. P

---

4. Q

for $\rightarrow$Intro

$\rightarrow$ Elim 1,3

Chain Argument

Example:

$P \rightarrow Q$

$Q \rightarrow R$

$P \rightarrow R$

1. $P \rightarrow Q$
2. $Q \rightarrow R$

---

3. P	for $\rightarrow$Intro
4. Q	$\rightarrow$ Elim 1,3
5. R	$\rightarrow$ Elim 2,4

Chain Argument

Example:

$P \rightarrow Q$

$Q \rightarrow R$

$P \rightarrow R$

1. $P \rightarrow Q$
2. $Q \rightarrow R$

---

3. P	for →Intro
4. Q	→ Elim 1,3
5. R	→ Elim 2,4
6. $P \rightarrow R$	→ Intro 3-5

Transitivity of →

Example:

$P \rightarrow Q$

$(Q \rightarrow R) \rightarrow (P \rightarrow R)$

1. $P \rightarrow Q$	
	$(Q \rightarrow R) \rightarrow (P \rightarrow R)$

Transitivity of →

Example:

$P \to Q$

$(Q \to R) \to (P \to R)$

1. $P \to Q$
2. $Q \to R$

$P \to R$ for →Intro

$(Q \to R) \to (P \to R)$ → Intro

Transitivity of →

Example:

| P → Q

| (Q→R)→(P→ R)

| 1. P → Q

| | 2. Q→R

| | | 3. P

| | | R

| | P → R → Intro

| (Q → R) → (P → R) → Intro

for →Intro

for →Intro

Transitivity of →

Example:

| $P \to Q$

| $(Q \to R) \to (P \to R)$

| 1. $P \to Q$

| | 2. $Q \to R$ — for →Intro

| | | 3. P — for →Intro

| | | 4. Q — → Elim 1,3

| | | R

| | $P \to R$ — → Intro

| $(Q \to R) \to (P \to R)$ — → Intro

TRANSITIVITY OF →

Example:

P → Q

(Q→R)→(P→ R)

1. P → Q

2. Q→R for →Intro

3. P for →Intro

4. Q → Elim 1,3

5. R → Elim 2,4

6. P → R → Intro 3-5

7. (Q → R) → (P → R) → Intro 2-6

Notice the Structure

1. P → Q
2. Q → R
   3. P — for →Intro
   4. Q — → Elim 1,3
   5. R — → Elim 2,4
6. P → R — → Intro 3-5

1. P → Q
2. Q → R — for →Intro
   3. P — for →Intro
   4. Q — → Elim 1,3
   5. R — → Elim 2,4
6. P → R — → Intro 3-5
7. (Q → R) → (P → R) — → Intro 2-6

SUBPROOFS AND PROOFS

P → Q

Q → R

P → R

SUBPROOFS AND PROOFS

P → Q

Q → R

P → R

SUBPROOFS AND PROOFS

$$\begin{array}{l} P \rightarrow Q \\ Q \rightarrow R \\ \hline P \rightarrow R \end{array}$$

$$\begin{array}{l} P \rightarrow Q \\ \hline (Q \rightarrow R) \rightarrow (P \rightarrow R) \end{array}$$

SUBPROOFS AND PROOFS

$P \rightarrow Q$

$Q \rightarrow R$

$P \rightarrow R$

$P \rightarrow Q$

$(Q \rightarrow R) \rightarrow (P \rightarrow R)$

$P \rightarrow Q$

$Q \rightarrow R$

P

R

SUBPROOFS AND PROOFS

$$\begin{array}{|l} P \rightarrow Q \\ Q \rightarrow R \\ \hline P \rightarrow R \end{array}$$

$$\begin{array}{|l} P \rightarrow Q \\ \hline (Q \rightarrow R) \rightarrow (P \rightarrow R) \end{array}$$

$$\begin{array}{|l} P \rightarrow Q \\ Q \rightarrow R \\ P \\ \hline R \end{array}$$

$$\begin{array}{|l} \\ \hline (P \rightarrow Q) \rightarrow [(Q \rightarrow R) \rightarrow (P \rightarrow R)] \end{array}$$