PROOFS WITH CONDITIONALS

Monday, 24 February

Conditional Elimination

- → Elimination: from P → Q and P, we can infer Q.

Conditional Elimination

- → Elimination: from P → Q and P, we can infer Q.

1.	P → Q	
2.	P	
3.	Q	→ Elim: 1,2

Conditional Elimination

- → Elimination: from P → Q and P, we can infer Q.

1. P → Q
2. P

---

3. Q → Elim: 1,2

1. (A ∨ B) → (C ∧ D)
2. A ∨ B

---

3. C ∧ D → Elim: 1,2

Example

Example:

$$\begin{array}{|l} A \\ A \rightarrow B \\ B \rightarrow C \\ \hline C \end{array}$$

Example

Example:

A
$A \rightarrow B$
$B \rightarrow C$

C

1. A
2. $A \rightarrow B$
3. $B \rightarrow C$

Example

Example:

A

$A \rightarrow B$

$B \rightarrow C$

C

1. A
2. $A \rightarrow B$
3. $B \rightarrow C$

---

4. B $\rightarrow$ Elim 1,2

Example

Example:

A
$A \rightarrow B$
$B \rightarrow C$

C

1. A
2. $A \rightarrow B$
3. $B \rightarrow C$

---

4. B $\rightarrow$ Elim 1,2
5. C $\rightarrow$ Elim 3,4

Example

Example:

$P \wedge Q$

$(P \vee R) \rightarrow S$

$(S \wedge Q) \rightarrow T$

T

Example

Example:

$P \wedge Q$
$(P \vee R) \rightarrow S$
$(S \wedge Q) \rightarrow T$

T

1. $P \wedge Q$
2. $(P \vee R) \rightarrow S$
3. $(S \wedge Q) \rightarrow T$

Example

Example:

$P \wedge Q$

$(P \vee R) \rightarrow S$

$(S \wedge Q) \rightarrow T$

T

1. $P \wedge Q$
2. $(P \vee R) \rightarrow S$
3. $(S \wedge Q) \rightarrow T$

---

4. P $\wedge$ Elim 1

Example

Example:

$P \wedge Q$

$(P \vee R) \rightarrow S$

$(S \wedge Q) \rightarrow T$

T

1. $P \wedge Q$
2. $(P \vee R) \rightarrow S$
3. $(S \wedge Q) \rightarrow T$

---

4. P — $\wedge$ Elim 1
5. Q — $\wedge$ Elim 1

Example

Example:

$P \wedge Q$
$(P \vee R) \rightarrow S$
$(S \wedge Q) \rightarrow T$

T

1. $P \wedge Q$
2. $(P \vee R) \rightarrow S$
3. $(S \wedge Q) \rightarrow T$

---

4. P $\wedge$ Elim 1
5. Q $\wedge$ Elim 1
6. $P \vee R$ $\vee$ Intro 4

Example

Example:

$P \wedge Q$
$(P \vee R) \rightarrow S$
$(S \wedge Q) \rightarrow T$

T

1. $P \wedge Q$
2. $(P \vee R) \rightarrow S$
3. $(S \wedge Q) \rightarrow T$

---

4. P $\wedge$ Elim 1
5. Q $\wedge$ Elim 1
6. $P \vee R$ $\vee$ Intro 4
7. S $\rightarrow$ Elim 2,6

Example

Example:

$P \wedge Q$
$(P \vee R) \rightarrow S$
$(S \wedge Q) \rightarrow T$

T

1. $P \wedge Q$
2. $(P \vee R) \rightarrow S$
3. $(S \wedge Q) \rightarrow T$

4.	P	$\wedge$ Elim 1
5.	Q	$\wedge$ Elim 1
6.	$P \vee R$	$\vee$ Intro 4
7.	S	$\rightarrow$ Elim 2,6
8.	$S \wedge Q$	$\wedge$ Intro 5,7

Example

Example:

$P \wedge Q$
$(P \vee R) \to S$
$(S \wedge Q) \to T$

T

1. $P \wedge Q$
2. $(P \vee R) \to S$
3. $(S \wedge Q) \to T$

4.	P	$\wedge$ Elim 1
5.	Q	$\wedge$ Elim 1
6.	$P \vee R$	$\vee$ Intro 4
7.	S	$\to$ Elim 2,6
8.	$S \wedge Q$	$\wedge$ Intro 5,7
9.	T	$\to$ Elim 3,8

Rules for Conditionals

Rules for Conditionals

1. $P \rightarrow Q$
2. P

---

3. Q $\rightarrow$ Elim: 1,2

Rules for Conditionals

1. $P \rightarrow Q$
2. P

---

3. Q $\rightarrow$ Elim: 1,2

1. $P \rightarrow Q$
2. $\neg P$

---

3. $\neg Q$ INVALID

Rules for Conditionals

1. $P \rightarrow Q$
2. P

---

3. Q $\rightarrow$ Elim: 1,2

1. $P \rightarrow Q$
2. $\neg P$

---

3. $\neg Q$ INVALID

1. $P \rightarrow Q$
2. Q

---

3. P INVALID

Rules for Conditionals

1. $P \rightarrow Q$
2. P

---

3. Q $\rightarrow$ Elim: 1,2

1. $P \rightarrow Q$
2. $\neg P$

---

3. $\neg Q$ INVALID

1. $P \rightarrow Q$
2. Q

---

3. P INVALID

1. $P \rightarrow Q$
2. $\neg Q$

---

3. $\neg P$ VALID, but not $\rightarrow$E

Rules for Conditionals

1. $P \rightarrow Q$
2. P

---

3. Q $\rightarrow$ Elim: 1,2

Modus Ponens

1. $P \rightarrow Q$
2. $\neg Q$

---

3. $\neg P$ VALID, but not $\rightarrow$E

Modus Tollens

Modus Tollens

Example:

1. $P \rightarrow Q$
2. $\neg Q$

---

3. $\neg P$

Modus Tollens

Example:

1. $P \rightarrow Q$
2. $\neg Q$

---

3. $\neg P$

1. $P \rightarrow Q$
2. $\neg Q$

$\neg P$

Modus Tollens

Example:

1. $P \rightarrow Q$
2. $\neg Q$

---

3. $\neg P$

1. $P \rightarrow Q$
2. $\neg Q$

---

3. P for $\neg$ Intro

$\neg P$ $\neg$ Intro

Modus Tollens

Example:

1. $P \rightarrow Q$
2. $\neg Q$

3. $\neg P$

1. $P \rightarrow Q$
2. $\neg Q$

3. P — for $\neg$ Intro
4. Q — $\rightarrow$ Elim 1,3

$\neg P$ — $\neg$ Intro

Modus Tollens

Example:

1. $P \rightarrow Q$
2. $\neg Q$

---

3. $\neg P$

1. $P \rightarrow Q$
2. $\neg Q$

---

3. P — for $\neg$ Intro

---

4. Q — $\rightarrow$ Elim 1,3
5. $\bot$ — $\bot$ Intro 2,4

$\neg P$ — $\neg$ Intro

Modus Tollens

Example:

1. $P \rightarrow Q$
2. $\neg Q$

---

3. $\neg P$

1. $P \rightarrow Q$	
2. $\neg Q$	
3. P	for $\neg$ Intro
4. Q	$\rightarrow$ Elim 1,3
5. $\bot$	$\bot$ Intro 2,4
6. $\neg P$	$\neg$ Intro 3-5

Biconditional Elimination

- ↔ Elimination: from P ↔ Q and P/Q, we can infer Q/P.

Biconditional Elimination

- ↔ Elimination: from P ↔ Q and P/Q, we can infer Q/P.

1. P ↔ Q	
2. P	
3. Q	↔ Elim: 1,2

BICONDITIONAL ELIMINATION

- ↔ Elimination: from P ↔ Q and P/Q, we can infer Q/P.

1. P ↔ Q	
2. P	
3. Q	↔ Elim: 1,2

1. P ↔ Q	
2. Q	
3. P	↔ Elim: 1,2

Example

Example:

A

$A \leftrightarrow B$

$\neg C \leftrightarrow B$

$\neg C$

Example

Example:

A

$A \leftrightarrow B$

$\neg C \leftrightarrow B$

$\neg C$

1. A
2. $A \leftrightarrow B$
3. $\neg C \leftrightarrow B$

Example

Example:

A

$A \leftrightarrow B$

$\neg C \leftrightarrow B$

$\neg C$

1. A
2. $A \leftrightarrow B$
3. $\neg C \leftrightarrow B$

---

4. B $\leftrightarrow$ Elim 1,2

Example

Example:

A
$A \leftrightarrow B$
$\neg C \leftrightarrow B$

$\neg C$

1. A	
2. $A \leftrightarrow B$	
3. $\neg C \leftrightarrow B$	
4. B	$\leftrightarrow$ Elim 1,2
5. $\neg C$	$\leftrightarrow$ Elim 3,4

Back to Case Arguments

Example:

$P \vee Q$

$P \rightarrow R$

$Q \rightarrow R$

R

Back to Case Arguments

Example:

$P \vee Q$

$P \rightarrow R$

$Q \rightarrow R$

R

1. $P \vee Q$
2. $P \rightarrow R$
3. $Q \rightarrow R$

R $\vee$Elim 1, 4-...

Back to Case Arguments

Example:

$P \vee Q$

$P \rightarrow R$

$Q \rightarrow R$

R

1. $P \vee Q$
2. $P \rightarrow R$
3. $Q \rightarrow R$

---

4. P — for $\vee$Elim

   R

   Q — for $\vee$Elim

   R

R — $\vee$Elim 1, 4-...

Back to Case Arguments

Example:

$P \vee Q$

$P \rightarrow R$

$Q \rightarrow R$

R

1. $P \vee Q$
2. $P \rightarrow R$
3. $Q \rightarrow R$

---

4. P — for ∨Elim

---

5. R — → Elim 2,4

6. Q — for ∨Elim

---

7. R — → Elim 3,6

8. R — ∨Elim 1, 4-5, 6-7

Back to Case Arguments

Example:

$P \vee Q$

$P \rightarrow R$

$Q \rightarrow R$

R

from P, infer R

Proof from P to R

1.	$P \vee Q$	
2.	$P \rightarrow R$	
3.	$Q \rightarrow R$	
4.	P	for $\vee$Elim
5.	R	$\rightarrow$ Elim 2,4
6.	Q	for $\vee$Elim
7.	R	$\rightarrow$ Elim 3,6
8.	R	$\vee$Elim 1, 4-5, 6-7

Subproofs and Conditionals

Subproofs and Conditionals

- The conditional $P \rightarrow R$ says that if P is true, then R is also true

Subproofs and Conditionals

- The conditional $P \rightarrow R$ says that if P is true, then R is also true
- A subproof from P to R shows that if P were true, then R would also be true.

Subproofs and Conditionals

- The conditional P→R says that if P is true, then R is also true
- A subproof from P to R shows that if P were true, then R would also be true.
- In $\mathcal{F}$, the disjunction elimination rule (∨Elim - case argument) uses a disjunction and two subproofs. In many other systems (and on Wikipedia for example) you need the disjunction and two conditionals

∨Elim (alternate version)

Example:

1. P ∨ Q	
2. P→R	
3. Q→R	
4. R	∨Elim 1, 2, 3

∨Elim (alternate version)

Example:

1. $P \vee Q$
2. $P \to R$
3. $Q \to R$

---

4. R ∨Elim 1, 2, 3

The missing subproofs <u>just are</u> the conditionals

Subproofs and Conditionals

- The conditional P→R says that if P is true, then R is also true
- A subproof from P to R shows that if P were true, then R would also be true.
- The Introduction rule for the conditional says that to prove that a conditional P→R is true, assume P and show that it does lead to R

Conditional Introduction

Conditional Introduction

- $\rightarrow$ Introduction
  From a subproof from P to Q, we can infer P $\rightarrow$ Q.

Conditional Introduction

- <u>→ Introduction</u>
  From a subproof from P to Q, we can infer P → Q.

```
| | 1. P
| |----
| | ...
| | j. Q
| k. P → Q          → Intro: 1-j
```

CONDITIONAL INTRODUCTION

- <u>→ Introduction</u>
  From a subproof from P to Q, we can infer P → Q.

| 1. P

| ...

| j. Q

k. P → Q → Intro: 1-j

This rule is often known as Conditional Proof

Chain Argument

Example:

$$P \rightarrow Q$$

$$Q \rightarrow R$$

$$P \rightarrow R$$

Chain Argument

Example:

$P \rightarrow Q$

$Q \rightarrow R$

$P \rightarrow R$

1. $P \rightarrow Q$
2. $Q \rightarrow R$

$P \rightarrow R$

Chain Argument

Example:

$P \rightarrow Q$

$Q \rightarrow R$

$P \rightarrow R$

1. $P \rightarrow Q$
2. $Q \rightarrow R$

---

3. P — for →Intro

R

$P \rightarrow R$ — →Intro

Chain Argument

Example:

$P \rightarrow Q$

$Q \rightarrow R$

$P \rightarrow R$

1. $P \rightarrow Q$
2. $Q \rightarrow R$

---

3. P for $\rightarrow$Intro

---

4. Q $\rightarrow$ Elim 1,3

R

$P \rightarrow R$ $\rightarrow$Intro

Chain Argument

Example:

$P \rightarrow Q$	
$Q \rightarrow R$	
$P \rightarrow R$	

1. $P \rightarrow Q$
2. $Q \rightarrow R$

3. P	for $\rightarrow$Intro
4. Q	$\rightarrow$ Elim 1,3
5. R	$\rightarrow$ Elim 2,4
$P \rightarrow R$	$\rightarrow$Intro

Chain Argument

Example:

$P \rightarrow Q$

$Q \rightarrow R$

$P \rightarrow R$

1.	$P \rightarrow Q$	
2.	$Q \rightarrow R$	
3.	P	for → Intro
4.	Q	→ Elim 1,3
5.	R	→ Elim 2,4
6.	$P \rightarrow R$	→ Intro 3-5

Chain Argument

Example:

$P \rightarrow Q$

$Q \rightarrow R$

$P \rightarrow R$

1. $P \rightarrow Q$
2. $Q \rightarrow R$
3. P — for $\rightarrow$Intro
4. Q — $\rightarrow$ Elim 1,3
5. R — $\rightarrow$ Elim 2,4
6. $P \rightarrow R$ — $\rightarrow$ Intro 3-5

A proof from P to R

Chain Argument

Example:

$P \rightarrow Q$

$Q \rightarrow R$

$P \rightarrow R$

1. $P \rightarrow Q$
2. $Q \rightarrow R$
3. P — for $\rightarrow$ Intro
4. Q — $\rightarrow$ Elim 1,3
5. R — $\rightarrow$ Elim 2,4
6. $P \rightarrow R$ — $\rightarrow$ Intro 3-5

A proof from P to R

So P leads to R

Import/Export Laws

Example:

$P \rightarrow (Q \rightarrow R)$

$(P \wedge Q) \rightarrow R$

Import/Export Laws

Example:

$P \rightarrow (Q \rightarrow R)$

$(P \wedge Q) \rightarrow R$

1. $P \rightarrow (Q \rightarrow R)$

$(P \wedge Q) \rightarrow R$

Import/Export Laws

Example:

$P \to (Q \to R)$

$(P \land Q) \to R$

1. $P \to (Q \to R)$

---

2. $P \land Q$ — for $\to$Intro

R

$(P \land Q) \to R$ — $\to$Intro

Import/Export Laws

Example:

$P \rightarrow (Q \rightarrow R)$

$(P \wedge Q) \rightarrow R$

1. $P \rightarrow (Q \rightarrow R)$

---

2. $P \wedge Q$ — for $\rightarrow$Intro

---

3. P — $\wedge$ Elim 2

R

$(P \wedge Q) \rightarrow R$ — $\rightarrow$Intro

Import/Export Laws

Example:

$P \rightarrow (Q \rightarrow R)$

$(P \wedge Q) \rightarrow R$

1. $P \rightarrow (Q \rightarrow R)$

---

2. $P \wedge Q$ — for $\rightarrow$Intro

---

3. P — $\wedge$ Elim 2
4. Q — $\wedge$ Elim 2

R

$(P \wedge Q) \rightarrow R$ — $\rightarrow$Intro

Import/Export Laws

Example:

| $P \to (Q \to R)$
|—
| $(P \wedge Q) \to R$

1. $P \to (Q \to R)$	
2. $P \wedge Q$	for $\to$Intro
3. P	$\wedge$ Elim 2
4. Q	$\wedge$ Elim 2
5. $Q \to R$	$\to$ Elim 1,3
R	
$(P \wedge Q) \to R$	$\to$Intro

Import/Export Laws

Example:

$P \to (Q \to R)$

$(P \land Q) \to R$

1.	$P \to (Q \to R)$	
2.	$P \land Q$	for $\to$Intro
3.	P	$\land$ Elim 2
4.	Q	$\land$ Elim 2
5.	$Q \to R$	$\to$ Elim 1,3
6.	R	$\to$ Elim 4,5
7.	$(P \land Q) \to R$	$\to$ Intro 2-6