

FAMOUS CASE ARGUMENT

Proof that it is possible for a, b to be irrational and a^b to be rational:

Is $\sqrt{2}^{\sqrt{2}}$ rational? I don't know, but...

If yes, then a, b, a^b are $a = \sqrt{2}, b = \sqrt{2}, a^b = \sqrt{2}^{\sqrt{2}}$

If no, then $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}, a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^2 = 2$

So either way, there is some a, b like this

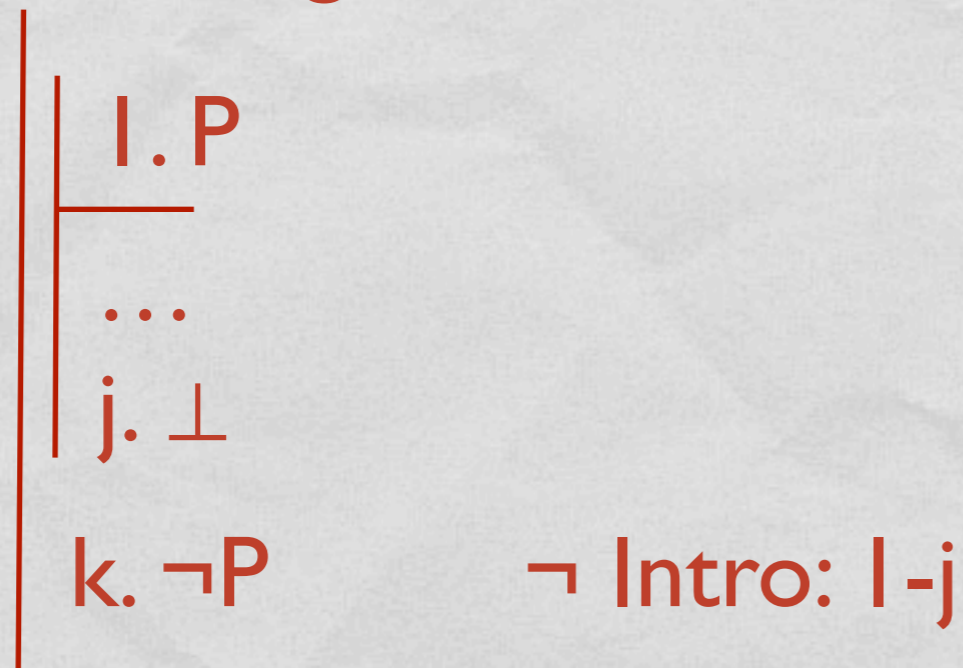
HARD PROOFS

Friday, 14 February

RULES USING CONTRADICTIONS

- \neg Introduction

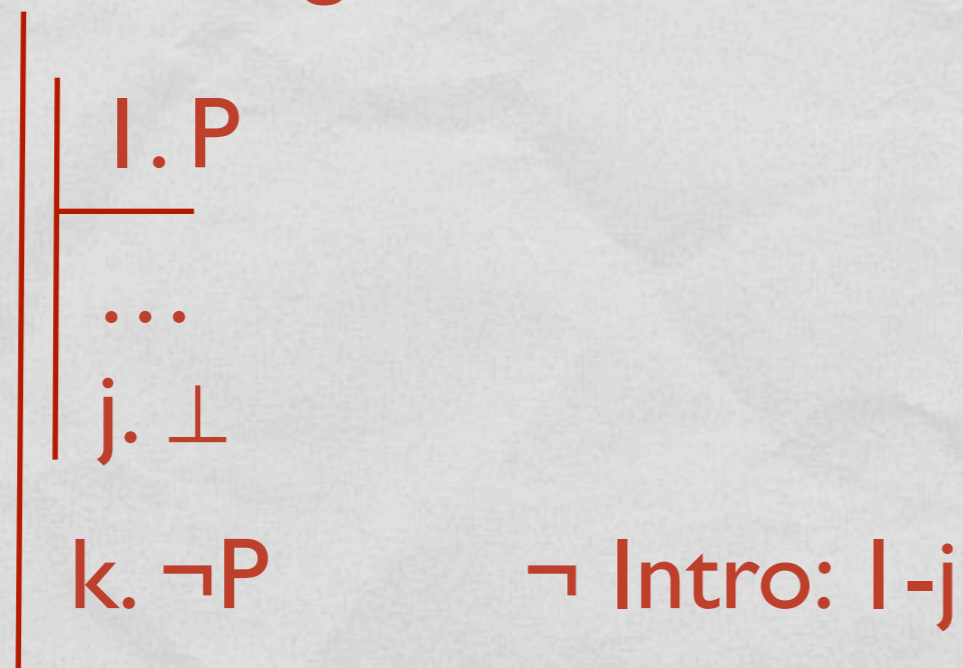
From showing P leads to \perp , we can infer $\neg P$.



RULES USING CONTRADICTIONS

- \neg Introduction

From showing P leads to \perp , we can infer $\neg P$.



- Within a subproof we derive \perp from P ;
outside the subproof we conclude $\neg P$.

RULES USING CONTRADICTIONS

- \perp Introduction
From P and $\neg P$, we can infer \perp .

RULES USING CONTRADICTIONS

- \perp Introduction

From P and $\neg P$, we can infer \perp .

	1. P	
	2. $\neg P$	
	<hr/>	
	3. \perp	\perp Intro: 1, 2

MORE \neg Intro

Example:

$a=b$

$\neg(\neg\text{Tet}(a) \wedge \text{Cube}(b))$

$\neg(\text{Small}(a) \wedge \text{Cube}(a))$

MORE \neg Intro

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$a=b$

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1. $a=b$

2. $\neg(\neg\text{Tet}(a) \wedge \text{Cube}(b))$

MORE \neg Intro

Example:

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MORE \neg Intro

Example:

	$a=b$
	$\neg(\neg\text{Tet}(a) \wedge \text{Cube}(b))$

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1. $a=b$

2. $\neg(\neg\text{Tet}(a) \wedge \text{Cube}(b))$

3. $\text{Small}(a) \wedge \text{Cube}(a)$ for \neg Intro

\perp

$\neg(\text{Small}(a) \wedge \text{Cube}(a))$ \neg Intro

MORE \neg Intro

Example:

$a=b$
$\neg(\neg\text{Tet}(a) \wedge \text{Cube}(b))$
$\neg(\text{Small}(a) \wedge \text{Cube}(a))$

1. $a=b$

2. $\neg(\neg\text{Tet}(a) \wedge \text{Cube}(b))$

3. $\text{Small}(a) \wedge \text{Cube}(a)$ for \neg Intro

4. $\text{Small}(a) \wedge \text{Cube}(a)$ elim 3

\perp

$\neg(\text{Small}(a) \wedge \text{Cube}(a))$ \neg Intro

MORE \neg Intro

Example:

$a=b$
$\neg(\neg\text{Tet}(a) \wedge \text{Cube}(b))$
$\neg(\text{Small}(a) \wedge \text{Cube}(a))$

1. $a=b$

2. $\neg(\neg\text{Tet}(a) \wedge \text{Cube}(b))$

3. $\text{Small}(a) \wedge \text{Cube}(a)$ for \neg Intro

4. $\text{Small}(a)$ \wedge elim 3

5. $\text{Cube}(a)$ \wedge elim 3

\perp

$\neg(\text{Small}(a) \wedge \text{Cube}(a))$ \neg Intro

MORE \neg Intro

Example:

$a=b$
$\neg(\neg\text{Tet}(a) \wedge \text{Cube}(b))$
<hr/>
$\neg(\text{Small}(a) \wedge \text{Cube}(a))$

1. $a=b$

2. $\neg(\neg\text{Tet}(a) \wedge \text{Cube}(b))$

3. $\text{Small}(a) \wedge \text{Cube}(a)$ for \neg Intro

4. $\text{Small}(a)$ \wedge elim 3

5. $\text{Cube}(a)$ \wedge elim 3

$\neg\text{Tet}(a) \wedge \text{Cube}(b)$

\perp

$\neg(\text{Small}(a) \wedge \text{Cube}(a))$ \neg Intro

MORE \neg Intro

Example:

$a=b$	
$\neg(\neg\text{Tet}(a) \wedge \text{Cube}(b))$	
<hr style="width: 50%; margin-left: 0;"/>	
$\neg(\text{Small}(a) \wedge \text{Cube}(a))$	

1. $a=b$

2. $\neg(\neg\text{Tet}(a) \wedge \text{Cube}(b))$

3. $\text{Small}(a) \wedge \text{Cube}(a)$ for \neg Intro

4. $\text{Small}(a)$ \wedge elim 3

5. $\text{Cube}(a)$ \wedge elim 3

6. $\neg\text{Tet}(a)$ Ana Con 5

$\neg\text{Tet}(a) \wedge \text{Cube}(b)$

\perp

$\neg(\text{Small}(a) \wedge \text{Cube}(a))$ \neg Intro

MORE \neg Intro

Example:

$a=b$	
$\neg(\neg\text{Tet}(a) \wedge \text{Cube}(b))$	
<hr style="width: 50%; margin-left: 0;"/>	
$\neg(\text{Small}(a) \wedge \text{Cube}(a))$	

1. $a=b$

2. $\neg(\neg\text{Tet}(a) \wedge \text{Cube}(b))$

3. $\text{Small}(a) \wedge \text{Cube}(a)$ for \neg Intro

4. $\text{Small}(a)$ \wedge elim 3

5. $\text{Cube}(a)$ \wedge elim 3

6. $\neg\text{Tet}(a)$ Ana Con 5

7. $\text{Cube}(b)$ = elim 1,5

$\neg\text{Tet}(a) \wedge \text{Cube}(b)$

\perp

$\neg(\text{Small}(a) \wedge \text{Cube}(a))$ \neg Intro

MORE \neg Intro

Example:

$a=b$
$\neg(\neg\text{Tet}(a) \wedge \text{Cube}(b))$
<hr/>
$\neg(\text{Small}(a) \wedge \text{Cube}(a))$

1. $a=b$
2. $\neg(\neg\text{Tet}(a) \wedge \text{Cube}(b))$
<hr/>
3. $\text{Small}(a) \wedge \text{Cube}(a)$
<hr/>
4. $\text{Small}(a)$ \wedge elim 3
5. $\text{Cube}(a)$ \wedge elim 3
6. $\neg\text{Tet}(a)$ Ana Con 5
7. $\text{Cube}(b)$ = elim 1,5
8. $\neg\text{Tet}(a) \wedge \text{Cube}(b)$ \wedge Intro 6,7
9. \perp \perp Intro 2,8
10. $\neg(\text{Small}(a) \wedge \text{Cube}(a))$ \neg Intro 3-9

PROOFS WITH NO PREMISES (6.35)

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- If the conclusion of an argument is a tautological consequence of the premises, then you can prove this conclusion from the premises in \mathcal{F} (this is known as *completeness*)

PROOFS WITH NO PREMISES (6.35)

- If the conclusion of an argument is a tautological consequence of the premises, then you can prove this conclusion from the premises in \mathcal{F} (this is known as *completeness*)
- Since tautologies are always true, they are tautological consequences of any set of premises - even the set of no premises at all!

PROOFS WITH NO PREMISES (6.35)

Example:

┌
└ $\neg(a=b \wedge \text{Large}(a) \wedge \text{Small}(b))$

PROOFS WITH NO PREMISES (6.35)

$$\neg(a=b \wedge \text{Large}(a) \wedge \text{Small}(b))$$

PROOFS WITH NO PREMISES (6.35)

1. $a=b \wedge \text{Large}(a) \wedge \text{Small}(b)$ for \neg Intro

2. $a=b$ \wedge elim 1

\perp

$\neg(a=b \wedge \text{Large}(a) \wedge \text{Small}(b))$ \neg Intro

PROOFS WITH NO PREMISES (6.35)

1. $a=b \wedge \text{Large}(a) \wedge \text{Small}(b)$ for \neg Intro

2. $a=b$ \wedge elim 1

3. $\text{Large}(a)$ \wedge elim 1

\perp

$\neg(a=b \wedge \text{Large}(a) \wedge \text{Small}(b))$ \neg Intro

PROOFS WITH NO PREMISES (6.35)

1. $a=b \wedge \text{Large}(a) \wedge \text{Small}(b)$ for \neg Intro

2. $a=b$ \wedge elim 1

3. $\text{Large}(a)$ \wedge elim 1

4. $\text{Small}(b)$ \wedge elim 1

\perp

$\neg(a=b \wedge \text{Large}(a) \wedge \text{Small}(b))$ \neg Intro

PROOFS WITH NO PREMISES (6.35)

1. $a=b \wedge \text{Large}(a) \wedge \text{Small}(b)$ for \neg Intro

2. $a=b$ \wedge elim 1

3. $\text{Large}(a)$ \wedge elim 1

4. $\text{Small}(b)$ \wedge elim 1

5. $\text{Large}(b)$ = elim 2, 3

\perp

$\neg(a=b \wedge \text{Large}(a) \wedge \text{Small}(b))$ \neg Intro

PROOFS WITH NO PREMISES (6.35)

1. $a=b \wedge \text{Large}(a) \wedge \text{Small}(b)$ for \neg Intro

2. $a=b$ \wedge elim 1

3. $\text{Large}(a)$ \wedge elim 1

4. $\text{Small}(b)$ \wedge elim 1

5. $\text{Large}(b)$ = elim 2, 3

6. \perp Ana Con 4,5

\perp

$\neg(a=b \wedge \text{Large}(a) \wedge \text{Small}(b))$ \neg Intro

PROOFS WITH NO PREMISES (6.35)

1. $a=b \wedge \text{Large}(a) \wedge \text{Small}(b)$ for \neg Intro

2. $a=b$ \wedge elim 1

3. $\text{Large}(a)$ \wedge elim 1

4. $\text{Small}(b)$ \wedge elim 1

5. $\text{Large}(b)$ = elim 2, 3

6. \perp Ana Con 4,5

7. $\neg(a=b \wedge \text{Large}(a) \wedge \text{Small}(b))$ \neg Intro 1-6

Compare to 6.11, 6.13

$P \vee Q \vee R$
$\neg P$
$\neg R$
Q

Compare to 6.11, 6.13

$$\begin{array}{l} P \vee Q \vee R \\ \neg P \\ \neg R \\ \hline Q \end{array}$$
$$\begin{array}{l} 1. P \vee Q \vee R \\ 2. \neg P \\ 3. \neg R \\ \hline \end{array}$$
$$Q$$

Compare to 6.11, 6.13

$P \vee Q \vee R$
$\neg P$
$\neg R$
Q

1. $P \vee Q \vee R$
2. $\neg P$
3. $\neg R$

4. $\neg Q$

for \neg Intro

\perp
Q

\neg Intro

Compare to 6.11, 6.13

$$\begin{array}{|l} P \vee Q \vee R \\ \neg P \\ \neg R \\ \hline Q \end{array}$$
$$\begin{array}{|l} 1. P \vee Q \vee R \\ 2. \neg P \\ 3. \neg R \\ \hline \end{array}$$
$$\begin{array}{|l} 4. \neg Q \\ \hline \end{array}$$

for \neg Intro

$$\begin{array}{|l} 5. P \\ \hline \end{array}$$

for \vee Elim

$$\perp$$

\vee Elim

$$Q$$

\neg Intro

Compare to 6.11, 6.13

	$P \vee Q \vee R$
	$\neg P$
	$\neg R$
	—
	Q

	1. $P \vee Q \vee R$
	2. $\neg P$
	3. $\neg R$
	—

		4. $\neg Q$
		—

for \neg Intro

			5. P
			—

for \vee Elim

			6. \perp
--	--	--	------------

\perp Intro 2,5

				\perp
--	--	--	--	---------

\vee Elim

				Q
--	--	--	--	-----

\neg Intro

Compare to 6.11, 6.13

$P \vee Q \vee R$
 $\neg P$
 $\neg R$
—
 Q

1. $P \vee Q \vee R$
2. $\neg P$
3. $\neg R$

4. $\neg Q$

for \neg Intro

5. P

for \vee Elim

6. \perp

\perp Intro 2,5

7. Q

for \vee Elim

\perp

\vee Elim

Q

\neg Intro

Compare to 6.11, 6.13

	$P \vee Q \vee R$
	$\neg P$
	$\neg R$
	—
	Q

	1. $P \vee Q \vee R$
	2. $\neg P$
	3. $\neg R$
	—

		4. $\neg Q$
		—

for \neg Intro

			5. P
			—

for \vee Elim

			6. \perp
			—

\perp Intro 2,5

			7. Q
			—

for \vee Elim

			8. \perp
			—

\perp Intro 4,7

			\perp
			—

\vee Elim

			Q
			—

\neg Intro

Compare to 6.11, 6.13

	$P \vee Q \vee R$
	$\neg P$
	$\neg R$
	—
	Q

	1. $P \vee Q \vee R$
	2. $\neg P$
	3. $\neg R$
	—

		4. $\neg Q$
		—

for \neg Intro

			5. P
			—

for \vee Elim

			6. \perp
			—

\perp Intro 2,5

			7. Q
			—

for \vee Elim

			8. \perp
			—

\perp Intro 4,7

			9. R
			—

for \vee Elim

				\perp
				—

\vee Elim

				Q
--	--	--	--	-----

\neg Intro

Compare to 6.11, 6.13

$P \vee Q \vee R$
$\neg P$
$\neg R$
Q

1. $P \vee Q \vee R$
2. $\neg P$
3. $\neg R$

4. $\neg Q$

for \neg Intro

5. P

for \vee Elim

6. \perp

\perp Intro 2,5

7. Q

for \vee Elim

8. \perp

\perp Intro 4,7

9. R

for \vee Elim

10. \perp

\perp Intro 3,9

\perp

\vee Elim

Q

\neg Intro

Compare to 6.11, 6.13

$P \vee Q \vee R$
$\neg P$
$\neg R$
<hr style="width: 100%;"/>
Q

1. $P \vee Q \vee R$
2. $\neg P$
3. $\neg R$
<hr style="width: 100%;"/>

4. $\neg Q$
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for \neg Intro

5. P
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for \vee Elim

6. \perp
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\perp Intro 2,5

7. Q
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for \vee Elim

8. \perp
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\perp Intro 4,7

9. R
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for \vee Elim

10. \perp
<hr style="width: 100%;"/>

\perp Intro 3,9

11. \perp
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\vee Elim 1,5-6,7-8,9-10

12. Q

\neg Intro 4-11

DOUBLE \neg Intro (6.30)

DOUBLE \neg Intro (6.30)

1. $\neg(\neg\text{Cube}(a) \wedge \text{Cube}(b))$
2. $\neg(\neg\text{Cube}(b) \vee \text{Cube}(c))$

Cube(a)

DOUBLE \neg Intro (6.30)

1. $\neg(\neg\text{Cube}(a) \wedge \text{Cube}(b))$
2. $\neg(\neg\text{Cube}(b) \vee \text{Cube}(c))$

3. $\neg\text{Cube}(a)$

for \neg Intro

$\text{Cube}(a)$

\neg Intro

DOUBLE \neg Intro (6.30)

1. $\neg(\neg\text{Cube}(a) \wedge \text{Cube}(b))$

2. $\neg(\neg\text{Cube}(b) \vee \text{Cube}(c))$

3. $\neg\text{Cube}(a)$

for \neg Intro

4. $\text{Cube}(b)$

for \neg Intro

$\text{Cube}(a)$

\neg Intro

DOUBLE \neg Intro (6.30)

1. $\neg(\neg\text{Cube}(a) \wedge \text{Cube}(b))$

2. $\neg(\neg\text{Cube}(b) \vee \text{Cube}(c))$

3. $\neg\text{Cube}(a)$ for \neg Intro

4. $\text{Cube}(b)$ for \neg Intro

5. $\neg\text{Cube}(a) \wedge \text{Cube}(b)$ \wedge Intro 3,4

$\text{Cube}(a)$

\neg Intro

DOUBLE \neg Intro (6.30)

1. $\neg(\neg\text{Cube}(a) \wedge \text{Cube}(b))$

2. $\neg(\neg\text{Cube}(b) \vee \text{Cube}(c))$

3. $\neg\text{Cube}(a)$ for \neg Intro

4. $\text{Cube}(b)$ for \neg Intro

5. $\neg\text{Cube}(a) \wedge \text{Cube}(b)$ \wedge Intro 3,4

6. \perp \perp Intro 1,5

$\text{Cube}(a)$

\neg Intro

DOUBLE \neg Intro (6.30)

1. $\neg(\neg\text{Cube}(a) \wedge \text{Cube}(b))$

2. $\neg(\neg\text{Cube}(b) \vee \text{Cube}(c))$

3. $\neg\text{Cube}(a)$ for \neg Intro

4. $\text{Cube}(b)$ for \neg Intro

5. $\neg\text{Cube}(a) \wedge \text{Cube}(b)$ \wedge Intro 3,4

6. \perp \perp Intro 1,5

7. $\neg\text{Cube}(b)$ \neg Intro 4-6

$\text{Cube}(a)$ \neg Intro

DOUBLE \neg Intro (6.30)

1. $\neg(\neg\text{Cube}(a) \wedge \text{Cube}(b))$

2. $\neg(\neg\text{Cube}(b) \vee \text{Cube}(c))$

3. $\neg\text{Cube}(a)$ for \neg Intro

4. $\text{Cube}(b)$ for \neg Intro

5. $\neg\text{Cube}(a) \wedge \text{Cube}(b)$ \wedge Intro 3,4

6. \perp \perp Intro 1,5

7. $\neg\text{Cube}(b)$ \neg Intro 4-6

8. $\neg\text{Cube}(b) \vee \text{Cube}(c)$ \vee Intro 7

$\text{Cube}(a)$ \neg Intro

DOUBLE \neg Intro (6.30)

1. $\neg(\neg\text{Cube}(a) \wedge \text{Cube}(b))$

2. $\neg(\neg\text{Cube}(b) \vee \text{Cube}(c))$

3. $\neg\text{Cube}(a)$ for \neg Intro

4. $\text{Cube}(b)$ for \neg Intro

5. $\neg\text{Cube}(a) \wedge \text{Cube}(b)$ \wedge Intro 3,4

6. \perp \perp Intro 1,5

7. $\neg\text{Cube}(b)$ \neg Intro 4-6

8. $\neg\text{Cube}(b) \vee \text{Cube}(c)$ \vee Intro 7

9. \perp \perp Intro 2,8

$\text{Cube}(a)$ \neg Intro

DOUBLE \neg Intro (6.30)

1. $\neg(\neg\text{Cube}(a) \wedge \text{Cube}(b))$

2. $\neg(\neg\text{Cube}(b) \vee \text{Cube}(c))$

3. $\neg\text{Cube}(a)$ for \neg Intro

4. $\text{Cube}(b)$ for \neg Intro

5. $\neg\text{Cube}(a) \wedge \text{Cube}(b)$ \wedge Intro 3,4

6. \perp \perp Intro 1,5

7. $\neg\text{Cube}(b)$ \neg Intro 4-6

8. $\neg\text{Cube}(b) \vee \text{Cube}(c)$ \vee Intro 7

9. \perp \perp Intro 2,8

10. $\text{Cube}(a)$ \neg Intro 3-8