Famous Case Argument

Proof that it is possible for *a*, *b* to be irrational and a^b to rational:

Is $\sqrt{2}^{\sqrt{2}}$ rational? I don't know, but...

If yes, then a, b, a^b are $a=\sqrt{2}, b=\sqrt{2}, a^b=\sqrt{2}^{\sqrt{2}}$

If no, then $a=\sqrt{2}^{\sqrt{2}}, b=\sqrt{2}, a^b=(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^2 = 2$

So either way, there is <u>some</u> *a*, *b* like this

Proofs in Boolean Logic

Wednesday, 12 February

CASE ARGUMENT

Example:

a=b

Cube(a) ∨ Tet(a)

Cube(b) ∨ Tet(b)

Case Argument

Example:

a=b
Cube(a) ∨ Tet(a)

Cube(b) ∨ Tet(b)

1. a=b
2. Cube(a) ∨ Tet(a)

Case Argument

Example:

a=b
Cube(a) ∨ Tet(a)

Cube(b) ∨ Tet(b)

1. a=b
2. Cube(a) ∨ Tet(a)

---

3. Cube(a) (for v-elim)

Case Argument

Example:

a=b
Cube(a) $\vee$ Tet(a)

Cube(b) $\vee$ Tet(b)

1. a=b
2. Cube(a) $\vee$ Tet(a)

---

3. Cube(a) (for v-elim)

---

4. Cube(b) = elim: 1,3

Case Argument

Example:

a=b

Cube(a) ∨ Tet(a)

Cube(b) ∨ Tet(b)

1. a=b
2. Cube(a) ∨ Tet(a)

---

3. Cube(a) (for v-elim)

---

4. Cube(b) = elim: 1,3
5. Cube(b) ∨ Tet(b) ∨ Intro: 4

Case Argument

Example:

a=b
Cube(a) ∨ Tet(a)

Cube(b) ∨ Tet(b)

1. a=b
2. Cube(a) ∨ Tet(a)

3. Cube(a) (for ∨-elim)

4. Cube(b) = elim: 1,3

5. Cube(b) ∨ Tet(b) ∨ Intro: 4

6. Tet(a) (for ∨-elim)

Case Argument

Example:

a=b
Cube(a) ∨ Tet(a)

Cube(b) ∨ Tet(b)

1. a=b
2. Cube(a) ∨ Tet(a)

---

3. Cube(a) (for v-elim)

---

4. Cube(b) = elim: 1,3
5. Cube(b) ∨ Tet(b) ∨ Intro: 4

6. Tet(a) (for v-elim)

---

7. Tet(b) = elim: 1,6

Case Argument

Example:

a=b

Cube(a) ∨ Tet(a)

Cube(b) ∨ Tet(b)

1. a=b
2. Cube(a) ∨ Tet(a)

---

3. Cube(a) (for ∨-elim)

---

4. Cube(b) = elim: 1,3
5. Cube(b) ∨ Tet(b) ∨ Intro: 4

6. Tet(a) (for ∨-elim)

---

7. Tet(b) = elim: 1,6
8. Cube(b) ∨ Tet(b) ∨ Intro: 7

Case Argument

Example:

a=b

Cube(a) ∨ Tet(a)

Cube(b) ∨ Tet(b)

1. a=b
2. Cube(a) ∨ Tet(a)

---

3. Cube(a) (for v-elim)

---

4. Cube(b) = elim: 1,3
5. Cube(b) ∨ Tet(b) ∨ Intro: 4

6. Tet(a) (for v-elim)

---

7. Tet(b) = elim: 1,6
8. Cube(b) ∨ Tet(b) ∨ Intro: 7

9. Cube(b) ∨ Tet(b) ∨Elim: 1,3-5,6-8

Example for you to do

Example:

A

$B \vee C$

$(A \wedge B) \vee (A \wedge C)$

Example for you to do

Example:

A

$B \vee C$

$(A \wedge B) \vee (A \wedge C)$

1. A
2. $B \vee C$

EXAMPLE FOR YOU TO DO

Example:

A

B ∨ C

(A ∧ B) ∨ (A ∧ C)

1. A
2. B ∨ C

---

3. B

(for v-elim)

Example for you to do

Example:

A

B ∨ C

(A ∧ B) ∨ (A ∧ C)

1. A
2. B ∨ C

3. B (for v-elim)

4. A ∧ B ∧ Intro: 1,3

Example for you to do

Example:

A
B ∨ C

(A ∧ B) ∨ (A ∧ C)

1. A
2. B ∨ C

3. B (for v-elim)

4. A ∧ B ∧ Intro: 1,3
5. (A ∧ B) ∨ (A ∧ C) ∨ Intro: 4

Example for you to do

Example:

A

B ∨ C

(A ∧ B) ∨ (A ∧ C)

1. A
2. B ∨ C

---

3. B (for v-elim)

---

4. A ∧ B ∧ Intro: 1,3
5. (A ∧ B) ∨ (A ∧ C) ∨ Intro: 4

6. C (for v-elim)

Example for you to do

Example:

A

$B \vee C$

$(A \wedge B) \vee (A \wedge C)$

1. A
2. $B \vee C$
3. B (for v-elim)
4. $A \wedge B$ $\wedge$ Intro: 1,3
5. $(A \wedge B) \vee (A \wedge C)$ $\vee$ Intro: 4
6. C (for v-elim)
7. $A \wedge C$ $\wedge$ Intro: 1,6

Example for you to do

Example:

$$
\begin{array}{|l}
A \\
B \lor C \\
\hline
(A \land B) \lor (A \land C)
\end{array}
$$

1. A
2. B ∨ C

3. B (for v-elim)
4. A ∧ B ∧ Intro: 1,3
5. (A ∧ B) ∨ (A ∧ C) ∨ Intro: 4

6. C (for v-elim)
7. A ∧ C ∧ Intro: 1,6
8. (A ∧ B) ∨ (A ∧ C) ∨ Intro: 7

Example for you to do

Example:

A

B ∨ C

(A ∧ B) ∨ (A ∧ C)

1. A
2. B ∨ C

---

3. B (for v-elim)

---

4. A ∧ B ∧ Intro: 1,3
5. (A ∧ B) ∨ (A ∧ C) ∨ Intro: 4

6. C (for v-elim)

---

7. A ∧ C ∧ Intro: 1,6
8. (A ∧ B) ∨ (A ∧ C) ∨ Intro: 7

9. (A∧B) ∨ (A∧C) ∨Elim: 1,3-5,6-8

Formal Proof Rules (¬)

- ¬ Elimination
  From ¬¬P, we can infer P.

Formal Proof Rules (¬)

- ¬ Elimination
  From ¬¬P, we can infer P.

1. ¬¬P	
2. P	¬ Elim: 1

Rules Using Contradictions

- ¬ Introduction
  From showing P leads to ⊥, we can infer ¬P.

	I. P	
	...	
	j. ⊥	
k. ¬P		¬ Intro: I-j

Rules Using Contradictions

- ¬ Introduction
  From showing P leads to ⊥, we can infer ¬P.

	| 1. P	
	|───	
	| …	
	| j. ⊥	
k. ¬P		¬ Intro: 1-j

- <u>Within a subproof</u> we derive ⊥ from P;
  <u>outside the subproof</u> we conclude ¬P.

Rules Using Contradictions

- ⊥ Introduction
  From P and ¬P, we can infer ⊥.

Rules Using Contradictions

- ⊥ Introduction
  From P and ¬P, we can infer ⊥.

1. P	
2. ¬P	
3. ⊥	⊥ Intro: 1, 2

Reductio Ad Absurdum

Example:

P

$\neg(P \land Q)$

$\neg Q$

Reductio Ad Absurdum

Example:

P

$\neg(P \wedge Q)$

$\neg Q$

1. P
2. $\neg(P \wedge Q)$

Reductio Ad Absurdum

Example:

P

¬(P ∧ Q)

¬Q

1. P

2. ¬(P ∧ Q)

---

3. Q

---

for ¬ Intro

Reductio Ad Absurdum

Example:

P

$\neg(P \wedge Q)$

$\neg Q$

1. P
2. $\neg(P \wedge Q)$

---

3. Q — for ¬ Intro

---

4. $P \wedge Q$ — ∧ Intro 1,3

Reductio Ad Absurdum

Example:

P

¬(P ∧ Q)

¬Q

1. P
2. ¬(P ∧ Q)

---

3. Q

---

4. P∧Q — ∧ Intro 1,3
5. ⊥ — ⊥ Intro 2,4

for ¬ Intro

REDUCTIO AD ABSURDUM

Example:

P

¬(P ∧ Q)

¬Q

1. P
2. ¬(P ∧ Q)

---

3. Q

---

4. P∧Q — ∧ Intro 1,3
5. ⊥ — ⊥ Intro 2,4

6. ¬Q — ¬ Intro 3-5

for ¬ Intro

Reductio Ad Absurdum

Example:

P

$\neg(P \wedge \neg Q)$

Q

Reductio Ad Absurdum

Example:

P

$\neg(P \wedge \neg Q)$

Q

1. P
2. $\neg(P \wedge \neg Q)$

Reductio Ad Absurdum

Example:

P

¬(P ∧ ¬Q)

Q

1. P
2. ¬(P ∧ ¬Q)

---

3. ¬Q

---

for ¬ Intro

Reductio Ad Absurdum

Example:

P

$\neg(P \wedge \neg Q)$

Q

1. P
2. $\neg(P \wedge \neg Q)$

---

3. $\neg Q$ — for $\neg$ Intro

---

4. $P \wedge \neg Q$ — $\wedge$ Intro 1,3

Reductio Ad Absurdum

Example:

P

¬(P ∧ ¬Q)

Q

1. P
2. ¬(P ∧ ¬Q)

---

3. ¬Q — for ¬ Intro

---

4. P∧¬Q — ∧ Intro 1,3
5. ⊥ — ⊥ Intro 2,4

Reductio Ad Absurdum

Example:

P

¬(P ∧ ¬Q)

Q

1. P
2. ¬(P ∧ ¬Q)

---

3. ¬Q — for ¬ Intro

---

4. P∧¬Q — ∧ Intro 1,3
5. ⊥ — ⊥ Intro 2,4

6. ¬¬Q — ¬ Intro 3-5

Reductio Ad Absurdum

Example:

P

¬(P ∧ ¬Q)

Q

1. P	
2. ¬(P ∧ ¬Q)	
3. ¬Q	for ¬ Intro
4. P∧¬Q	∧ Intro 1,3
5. ⊥	⊥ Intro 2,4
6. ¬¬Q	¬ Intro 3-5
7. Q	¬ Elim 6

Reductio Ad Absurdum

1. P	
2. $\neg(P \wedge Q)$	
3. Q	for $\neg$ Intro
4. $P \wedge Q$	$\wedge$ Intro 1,3
5. $\bot$	$\bot$ Intro 2,4
6. $\neg Q$	$\neg$ Intro 3-5

1. P	
2. $\neg(P \wedge \neg Q)$	
3. $\neg Q$	for $\neg$ Intro
4. $P \wedge \neg Q$	$\wedge$ Intro 1,3
5. $\bot$	$\bot$ Intro 2,4
6. $\neg\neg Q$	$\neg$ Intro 3-5
7. Q	$\neg$ Elim 6

For you to try

Example:

$\neg(a=b \vee b=c)$

$a \neq b$

1. $\neg(a=b \vee b=c)$

For you to try

Example:

$\neg(a=b \lor b=c)$

$a \neq b$

1. $\neg(a=b \lor b=c)$

---

2. $a=b$

FOR YOU TO TRY

Example:

$\neg(a=b \lor b=c)$

$a \neq b$

1. $\neg(a=b \lor b=c)$
2. $a=b$
3. $a=b \lor b=c$ $\lor$ Intro 2

For you to try

Example:

$\neg(a=b \vee b=c)$

$a \neq b$

1. $\neg(a=b \vee b=c)$
2. $a=b$
3. $a=b \vee b=c$ — $\vee$ Intro 2
4. $\bot$ — $\bot$ Intro 1,3

FOR YOU TO TRY

Example:

| ¬(a=b ∨ b=c)
|———
| a≠b

1. ¬(a=b ∨ b=c)	
2. a=b	
3. a=b ∨ b=c	∨ Intro 2
4. ⊥	⊥ Intro 1,3
5. a≠b	¬ Intro 2-4

More ¬Elim

Example:

a=b

¬(¬Tet(a) ∧ Cube(b))

¬(Small(a) ∧ Cube(a))

1. a=b
2. ¬(¬Tet(a) ∧ Cube(b))

More ¬Elim

Example:

a=b

¬(¬Tet(a) ∧ Cube(b))

¬(Small(a) ∧ Cube(a))

1. a=b
2. ¬(¬Tet(a) ∧ Cube(b))

3. Small(a) ∧ Cube(a)

More ¬Elim

Example:

a=b

¬(¬Tet(a) ∧ Cube(b))

¬(Small(a) ∧ Cube(a))

1. a=b
2. ¬(¬Tet(a) ∧ Cube(b))

---

3. Small(a) ∧ Cube(a)

---

4. Small(a) ∧ elim 3

More ¬Elim

Example:

a=b

¬(¬Tet(a) ∧ Cube(b))

¬(Small(a) ∧ Cube(a))

1. a=b
2. ¬(¬Tet(a) ∧ Cube(b))

---

3. Small(a) ∧ Cube(a)

---

4. Small(a) ∧ elim 3
5. Cube(a) ∧ elim 3

More ¬Elim

Example:

a=b
¬(¬Tet(a) ∧ Cube(b))

¬(Small(a) ∧ Cube(a))

1. a=b
2. ¬(¬Tet(a) ∧ Cube(b))

---

3. Small(a) ∧ Cube(a)

---

4. Small(a) ∧ elim 3
5. Cube(a) ∧ elim 3
6. ¬Tet(a) Ana Con 5

More ¬Elim

Example:

a=b

¬(¬Tet(a) ∧ Cube(b))

¬(Small(a) ∧ Cube(a))

1. a=b
2. ¬(¬Tet(a) ∧ Cube(b))

---

3. Small(a) ∧ Cube(a)

---

4. Small(a) ∧ elim 3
5. Cube(a) ∧ elim 3
6. ¬Tet(a) Ana Con 5
7. Cube(b) = elim 1,5

More ¬Elim

Example:

a=b

¬(¬Tet(a) ∧ Cube(b))

¬(Small(a) ∧ Cube(a))

1. a=b
2. ¬(¬Tet(a) ∧ Cube(b))
3. Small(a) ∧ Cube(a)
4. Small(a) ∧ elim 3
5. Cube(a) ∧ elim 3
6. ¬Tet(a) Ana Con 5
7. Cube(b) = elim 1,5
8. ¬Tet(a) ∧ Cube(b) ∧Intro 6,7

More ¬Elim

Example:

| a=b
| ¬(¬Tet(a) ∧ Cube(b))
|———
| ¬(Small(a) ∧ Cube(a))

1.	a=b	
2.	¬(¬Tet(a) ∧ Cube(b))	
3.	Small(a) ∧ Cube(a)	
4.	Small(a)	∧ elim 3
5.	Cube(a)	∧ elim 3
6.	¬Tet(a)	Ana Con 5
7.	Cube(b)	= elim 1,5
8.	¬Tet(a) ∧ Cube(b)	∧Intro 6,7
9.	⊥	⊥ Intro 2,8

More ¬Elim

Example:

| a=b
| ¬(¬Tet(a) ∧ Cube(b))
|---
| ¬(Small(a) ∧ Cube(a))

1.	a=b	
2.	¬(¬Tet(a) ∧ Cube(b))	
3.	Small(a) ∧ Cube(a)	
4.	Small(a)	∧ elim 3
5.	Cube(a)	∧ elim 3
6.	¬Tet(a)	Ana Con 5
7.	Cube(b)	= elim 1,5
8.	¬Tet(a) ∧ Cube(b)	∧Intro 6,7
9.	⊥	⊥ Intro 2,8
10.	¬(Small(a) ∧ Cube(a))	¬ Intro 3-9

Rules Using Contradictions

- ⊥ Elimination
  From ⊥, we can infer absolutely whatever we want.

- This is helpful when we want to eliminate a disjunct when we know that its negation is true.
- We don't technically need this rule; we could just use ¬ Intro and ¬ Elim.

Rules Using Contradictions

- ⊥ Elimination

  From ⊥, we can infer absolutely whatever we want.

  | 1. ⊥
  |———
  | 2. BlueCheese(Moon) ⊥ Elim: 1

- This is helpful when we want to eliminate a disjunct when we know that its negation is true.
- We don't technically need this rule; we could just use ¬ Intro and ¬ Elim.

Rules Using Contradictions

- $\bot$ Elimination

  From $\bot$, we can infer absolutely whatever we want.

1.	$\bot$	
2.	BlueCheese(Moon)	$\bot$ Elim: 1

- This is helpful when we want to eliminate a disjunct when we know that its negation is true.

Rules Using Contradictions

- ⊥ Elimination

  From ⊥, we can infer absolutely whatever we want.

  | 1. ⊥
  |---
  | 2. BlueCheese(Moon)   ⊥ Elim: 1

- This is helpful when we want to eliminate a disjunct when we know that its negation is true.
- We don't technically need this rule; we could just use ¬ Intro and ¬ Elim.

Rules Using Contradictions

Example: Disjunctive Syllogism

$P \vee Q$

$\neg P$

Q

Rules Using Contradictions

Example: Disjunctive Syllogism

$P \vee Q$

$\neg P$

Q

1. $P \vee Q$
2. $\neg P$

Rules Using Contradictions

Example: Disjunctive Syllogism

$P \vee Q$
$\neg P$
Q

1. $P \vee Q$
2. $\neg P$
   3. P

Rules Using Contradictions

Example: Disjunctive Syllogism

| P ∨ Q
| ¬P
|———
| Q

1. P ∨ Q
2. ¬P
3. P
4. ⊥ ⊥ Intro 2,3

Rules Using Contradictions

Example: Disjunctive Syllogism

| P ∨ Q
| ¬P
|---
| Q

1. P ∨ Q
2. ¬P
3. P
4. ⊥	⊥ Intro 2,3
5. Q	⊥ Elim 4

Rules Using Contradictions

Example: Disjunctive Syllogism

$P \lor Q$

$\neg P$

Q

1. $P \lor Q$
2. $\neg P$

---

3. P

---

4. $\bot$ — $\bot$ Intro 2,3
5. Q — $\bot$ Elim 4

6. Q

Rules Using Contradictions

Example: Disjunctive Syllogism

| P ∨ Q
| ¬P
|———
| Q

1. P ∨ Q
2. ¬P	
3. P	
4. ⊥	⊥ Intro 2,3
5. Q	⊥ Elim 4
6. Q	
7. Q	∨ Elim 1,3-5,6-6