Puzzle

You meet three individuals - one knight who always tells the truth, a knave who always lies, and a normal person who can do either. They know each other's identities.

A says "B is the normal one."

B says "No, C is the normal one."

C says "No, B is definitely the normal one."

Who is what?

Reductio Ad Absurdum

Monday, 10 February

PROOF BY CASES

- ∨ Elimination

  If R follows from P, and if R follows from Q, then from P∨Q, we can infer R.

1. P ∨ Q

2. P

…

j. R ??

k. Q

…

m. R ??

n. R ∨Elim: 1,2-j,k-m

Scope Lines

Scope Lines indicate assumptions that don't necessarily follow from earlier assumptions

NESTED SUBPROOFS

Nested Subproofs

- We can introduce any assumption you want any time in a proof by introducing a new scope line. (If you do so, make sure you know how to get rid of it).

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NESTED SUBPROOFS

- We can introduce any assumption you want any time in a proof by introducing a new scope line. (If you do so, make sure you know how to get rid of it).
- Some proofs require nested subproofs - subproofs inside other subproofs.
- Example - when you have two ∨Elims in the same proof.

NESTED SUBPROOFS

$P \vee Q$

$R \vee S$

$(P \wedge R) \vee (P \wedge S) \vee (Q \wedge R) \vee (Q \wedge S)$

Download the complete proof done in Fitch from the website

Formal Proof Rules (¬)

- ¬ Elimination
  From ¬¬P, we can infer P.

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  From ¬¬P, we can infer P.

1. ¬¬P	
2. P	¬ Elim: 1

FORMAL PROOF RULES (¬)

- ¬ Elimination
  From ¬¬P, we can infer P.

1. ¬¬P	
2. P	¬ Elim: 1

- **Incorrect** (not main connective)

1. ¬¬P ∨ Q	
2. P ∨ Q	¬ Elim: 1

Formal Proof Rules (¬)

- ¬ Elimination
  From ¬¬P, we can infer P.

1.	¬¬(P→(Q↔R))	
2.	P→(Q↔R)	¬ Elim: 1

Formal Proof Rules (¬)

- ¬ Introduction
  This is our rule that formalizes the proof technique known as indirect proof, or Reductio Ad Absurdum. To prove something, assume it is false and show that this leads to contradiction.

Boring Reductios

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- I know you must have a high school degree. If you didn't, you couldn't be enrolled at Texas Tech. But here you are.

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- I know you must have a high school degree. If you didn't, you couldn't be enrolled at Texas Tech. But here you are.

- I didn't do laundry yesterday. If I did, I wouldn't have this giant pile of dirty laundry in my hamper.

Famous Reductios

- $\sqrt{2}$ must be irrational. If it were rational, it would be equal to p/q where p and q are integers. But then $(p/q)^2 = p^2/q^2 = (\sqrt{2})^2 = 2$

  So $p^2 = 2q^2$ and so p^2 is even and so p is even
  But then $p = 2n$ for some n and so $p^2 = (2n)^2 = 4n^2$ and so $4n^2 = 2q^2$ and so q is also even. But if p and q are both even, then p/q is not in lowest terms. Contradiction. So $\sqrt{2}$ can't be rational so it is irrational.

Famous Reductios

Famous Reductios

- There are an infinite number of prime numbers. If there weren't, there would be a greatest one. Call it P. Now take all the primes less than or equal to P and multiply them together and add 1. Call this X. If X is prime, it is bigger than P (since P is a factor). If X is not prime, it has prime factors bigger than P (since none of the primes P or less could be factors of both P and P+1) ...

Famous Reductios

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- $\sqrt{2}$ must be irrational. If it were rational, it would be equal to p/q where p and q are integers. But... (see chap 4)

Famous Reductios

- $\sqrt{2}$ must be irrational. If it were rational, it would be equal to p/q where p and q are integers. But... (see chap 4)
- There are an infinite number of prime numbers. If there weren't, there would be a greatest one. Call it P. Now take all the primes less than or equal to P and multiply them together and add 1. Call this X. If X is prime, it is bigger than P. If X is not prime, it has prime factors bigger than P...

Rules Using Contradictions

- ¬ Introduction
  From showing P leads to ⊥, we can infer ¬P.

1. P	
...	
j. ⊥	
k. ¬P	¬ Intro: 1-j

Rules Using Contradictions

- ¬ Introduction
  From showing P leads to ⊥, we can infer ¬P.

	| 1. P	
	| ...	
	| j. ⊥	
k. ¬P		¬ Intro: 1-j

- Within a subproof we derive ⊥ from P;
  outside the subproof we conclude ¬P.

Contradictions

- We use the special symbol $\bot$ to represent a contradiction. This sentence is always false - it is false on every row of any table.

Contradictions

- We use the special symbol ⊥ to represent a contradiction. This sentence is always false - it is false on every row of any table.

- This means that

  ⊥ is tautologically equivalent to $P \wedge \neg P$

Rules Using Contradictions

- $\perp$ Introduction
  From P and $\neg$P, we can infer $\perp$.

Rules Using Contradictions

- ⊥ Introduction
  From P and ¬P, we can infer ⊥.

1. P		
2. ¬P		
3. ⊥		⊥ Intro: 1, 2

Rules Using Contradictions

- ⊥ Introduction
  From P and ¬P, we can infer ⊥.

1. Smaller(a,b) ∨ Cube(b)	
2. ¬(Smaller(a,b) ∨ Cube(b))	
3. ⊥	⊥ Intro: 1, 2

Reductio Ad Absurdum

Example:

| P
| $\neg(P \wedge Q)$
|―――――
| $\neg Q$

Reductio Ad Absurdum

Example:

P

$\neg(P \wedge Q)$

$\neg Q$

1. P
2. $\neg(P \wedge Q)$

Reductio Ad Absurdum

Example:

P

$\neg(P \wedge Q)$

$\neg Q$

1. P
2. $\neg(P \wedge Q)$

Reductio Ad Absurdum

Example:

P

¬(P ∧ Q)

¬Q

1. P
2. ¬(P ∧ Q)

---

3. Q

---

for ¬ Intro

Reductio Ad Absurdum

Example:

P

$\neg(P \wedge Q)$

$\neg Q$

1. P
2. $\neg(P \wedge Q)$

---

3. Q

---

4. $P \wedge Q$ — $\wedge$ Intro 1,3

for $\neg$ Intro

Reductio Ad Absurdum

Example:

P

$\neg(P \land Q)$

$\neg Q$

1. P
2. $\neg(P \land Q)$

   3. Q — for $\neg$ Intro
   4. $P \land Q$ — $\land$ Intro 1,3
   5. $\bot$ — $\bot$ Intro 2,4

Reductio Ad Absurdum

Example:

P

¬(P ∧ Q)

¬Q

1. P
2. ¬(P ∧ Q)

---

3. Q — for ¬ Intro

---

4. P∧Q — ∧ Intro 1,3
5. ⊥ — ⊥ Intro 2,4

6. ¬Q — ¬ Intro 3-5

Reductio Ad Absurdum

Example:

$\neg(a=b \vee b=c)$

$a \neq b \wedge b \neq c$

1. $\neg(a=b \vee b=c)$

Reductio Ad Absurdum

Example:

$\neg(a=b \vee b=c)$

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1. $\neg(a=b \vee b=c)$

2. $a=b$

Reductio Ad Absurdum

Example:

$\neg(a=b \vee b=c)$

$a \neq b \wedge b \neq c$

1. $\neg(a=b \vee b=c)$

2. $a=b$

3. $a=b \vee b=c$ $\quad$ $\vee$ Intro 2

Reductio Ad Absurdum

Example:

$\neg(a=b \vee b=c)$

$a \neq b \wedge b \neq c$

1. $\neg(a=b \vee b=c)$
2. $a=b$
3. $a=b \vee b=c$ $\quad \vee$ Intro 2
4. $\bot$ $\quad \bot$ Intro 1,3

Reductio Ad Absurdum

Example:

$\neg(a=b \lor b=c)$

$a \neq b \land b \neq c$

1.	$\neg(a=b \lor b=c)$	
2.	$a=b$	
3.	$a=b \lor b=c$	$\lor$ Intro 2
4.	$\bot$	$\bot$ Intro 1,3
5.	$a \neq b$	$\neg$ Intro 2-4

Reductio Ad Absurdum

Example:

$\neg(a=b \lor b=c)$

$a\neq b \land b\neq c$

1. $\neg(a=b \lor b=c)$
2. $a=b$
3. $a=b \lor b=c$ — $\lor$ Intro 2
4. $\bot$ — $\bot$ Intro 1,3
5. $a\neq b$ — $\neg$ Intro 2-4
6. $b=c$

Reductio Ad Absurdum

Example:

$\neg(a=b \lor b=c)$

$a \neq b \land b \neq c$

1. $\neg(a=b \lor b=c)$	
2. $a=b$	
3. $a=b \lor b=c$	$\lor$ Intro 2
4. $\bot$	$\bot$ Intro 1,3
5. $a \neq b$	$\neg$ Intro 2-4
6. $b=c$	
7. $a=b \lor b=c$	$\lor$ Intro 6

Reductio Ad Absurdum

Example:

$\neg(a=b \vee b=c)$

$a \neq b \wedge b \neq c$

1.	$\neg(a=b \vee b=c)$	
2.	$a=b$	
3.	$a=b \vee b=c$	$\vee$ Intro 2
4.	$\bot$	$\bot$ Intro 1,3
5.	$a \neq b$	$\neg$ Intro 2-4
6.	$b=c$	
7.	$a=b \vee b=c$	$\vee$ Intro 6
8.	$\bot$	$\bot$ Intro 1,7

Reductio Ad Absurdum

Example:

$\neg(a=b \vee b=c)$

$a \neq b \wedge b \neq c$

1.	$\neg(a=b \vee b=c)$	
2.	$a=b$	
3.	$a=b \vee b=c$	$\vee$ Intro 2
4.	$\bot$	$\bot$ Intro 1,3
5.	$a \neq b$	$\neg$ Intro 2-4
6.	$b=c$	
7.	$a=b \vee b=c$	$\vee$ Intro 6
8.	$\bot$	$\bot$ Intro 1,7
9.	$b \neq c$	$\neg$ Intro 6-8

Reductio Ad Absurdum

Example:

$\neg(a=b \vee b=c)$

$a \neq b \wedge b \neq c$

1.	$\neg(a=b \vee b=c)$	
2.	$a=b$	
3.	$a=b \vee b=c$	$\vee$ Intro 2
4.	$\bot$	$\bot$ Intro 1,3
5.	$a \neq b$	$\neg$ Intro 2-4
6.	$b=c$	
7.	$a=b \vee b=c$	$\vee$ Intro 6
8.	$\bot$	$\bot$ Intro 1,7
9.	$b \neq c$	$\neg$ Intro 6-8
10.	$a \neq b \wedge b \neq c$	$\wedge$ Intro 5-9

Rules Using Contradictions

- ⊥ Elimination
  From ⊥, we can infer absolutely whatever we want.

- This is helpful when we want to eliminate a disjunct when we know that its negation is true.
- We don't technically need this rule; we could just use ¬ Intro and ¬ Elim.

Rules Using Contradictions

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  From ⊥, we can infer absolutely whatever we want.

1. ⊥	
2. BlueCheese(Moon)	⊥ Elim: 1

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Rules Using Contradictions

Example: Disjunctive Syllogism

$$\begin{array}{|l} P \lor Q \\ \neg P \\ \hline Q \end{array}$$

Rules Using Contradictions

Example: Disjunctive Syllogism

$P \vee Q$

$\neg P$

Q

1. $P \vee Q$
2. $\neg P$

Rules Using Contradictions

Example: Disjunctive Syllogism

$P \vee Q$

$\neg P$

Q

1. $P \vee Q$
2. $\neg P$

---

3. P

Rules Using Contradictions

Example: Disjunctive Syllogism

| $P \lor Q$
| $\neg P$
|---
| Q

1. $P \lor Q$
2. $\neg P$

---

3. P

---

4. $\bot$ — $\bot$ Intro 2,3

Rules Using Contradictions

Example: Disjunctive Syllogism

| P ∨ Q
| ¬P
|---
| Q

| 1. P ∨ Q |
2. ¬P
	3. P
|---
	4. ⊥ ⊥ Intro 2,3
	5. Q ⊥ Elim 4

Rules Using Contradictions

Example: Disjunctive Syllogism

$P \vee Q$

$\neg P$

Q

1. $P \vee Q$
2. $\neg P$

---

3. P

---

4. $\bot$ — $\bot$ Intro 2,3
5. Q — $\bot$ Elim 4

6. Q

Rules Using Contradictions

Example: Disjunctive Syllogism

$P \vee Q$
$\neg P$
Q

1.	$P \vee Q$	
2.	$\neg P$	
3.	P	
4.	$\bot$	$\bot$ Intro 2,3
5.	Q	$\bot$ Elim 4
6.	Q	
7.	Q	$\vee$ Elim 1,3-5,6-6