Handout for HPS 122

Definition: A gamble on A made at odds of a:b for stakes of $S is such that it pays $bS/(a+b) if A is true and –$aS/(a+b) if A is false. (i.e. if you are betting with a bookie, you pay $a to place a bet where you would win back $(a+b) and net $b in winnings if you win the bet.) Since the stakes are arbitrary, odds of a:b are the same as odds of ax : bx for any x.

Definition: A conditional gamble on A given B at odds of a:b at stakes $S is such that it pays $bS/(a+b) if A&B is true, pays –$aS/(a+b) if (~A)&B, and pays 0 if ~B (i.e. the bet is called off).

Definition: An agent considers a gamble fair if the subjective expected utility of the gamble is $0.

Definition: An agent's betting quotient (or Degree of Belief) for a proposition A [abbreviated DoB(A)]is the value q such that the agent considers a bet at price $q for stake $1 to be fair.

-- Assuming that the agent values money linearly, [so that U($x) + U($y) = U($x+y)], this means that if the agent considers a bet on A at odds a:b fair, then the agent's betting quotient is a/(a+b).

Conditional betting:

Lets call DoB(A&B) = q
Lets call DoB(B) = r
For simplicity and clarity, we can assume here DoB(~B) = 1-r

A conditional bet on A given B at price $DoB(A&B)/DoB(B) to win $1 has the following payoff table:

	Total payoff in dollars = winnings minus amount paid for bet
A & B	1-(q/r)
A & ~B	0
~A & B	-q/r
~A & ~B	0

We can simulate a conditional bet on A given B by making two unconditional bets – one on A&B and one on ~B.

In order to come up with the two bets, we aim to have the same payoff table as above. We are going to make a bet on A&B and a bet on ~B. The easiest way to make sure we have the right payoff table is to aim to have 0 total payoff in the ~B cases.

Since we will lose the A&B bet, we make sure that the total net winnings on the ~B bet exactly cancel out the cost of placing the A&B bet.

The following theorem makes this easier to see how to do:
Theorem: P(A|B) = P(A&B) + P(~B) x P(A&B)/P(B)

Thus if we wanted to simulate a bet on A given B with stakes of $1, we could use a bet on A&B stakes of $1 and a bet on ~B at stakes of $DoB(A&B)/DoB(B) = $q/r

Bet 1: Bet on A&B at $q to win $1 – this is odds q:1-q
Bet 2: Bet on ~B at $(q/r) x (1-r) to win $q/r – this is odds (q/r) x (1-r) : (q/r) – (q/r) x (1-r) which is the same as odds (1-r) : r

Now the results of making these two bets together:

	Payoff bet 1	Payoff bet 2	Total Payoff
A & B	1-q	-(q/r) x (1-r)	1-(q/r)
A & ~B	-q	(q/r) – [(q/r) x (1-r)] = q	0
~A & B	-q	-(q/r) x (1-r)	-q/r
~A & ~B	-q	(q/r) – [(q/r) x (1-r)] = q	0

Notice that the payoff table for making both of these bets is exactly the same as making a single conditional bet. Therefore, if you value the single conditional bet at a different rate than you value the sum of these two bets, you can be Dutch Booked.