Monty Hall Problem
Handout for HPS 122

Wikipedia has a very lengthy article with some interesting history and lots of details about the Monty Hall Problem. It is well worth reading.

http://en.wikipedia.org/wiki/Monty_Hall_problem

I said in class that under certain assumptions (Monty always opens an empty door and chooses randomly if he has a choice) you haven't learned anything relevant and so your probabilities shouldn't change. This is of course false. You have learned at minimum that the prize is not behind door 2 and so your probabilities should change. What I meant and should have said was that your probability for door 1 should not change. But of course door 2 should go to zero and door 3 should go to 2/3.

It is very instructive to put the problem in the following way. Lets say that what we learned was that the prize was not behind door 2. In that case, conditionalizing on ~D2 will lead us to P(D1) = 1/2. That is, P(D1|~D2) = 1/2. Part of understanding the answer is figuring out how you learn anything relevant beyond that.

Of course what you really learned is that Monty opened door 2 after you selected door 1. But why is conditioning on this different than conditioning on just ~D2? Because there are different scenarios in which he could open door 2 and the probabilities and likelihoods for these scenarios are different.

This is what you should believe:

Lets call the hypothesis that the prize is behind door 1 D1. Call the fact that Monty revealed door 2 to be empty after you chose door 1 OD2.

You have selected door 1 and Monty has not yet revealed anything. We will call this t1 at which point you have some probability function P_1. Now after t1 Monty then reveals that door 2 is empty. Now you have function P_2. We assume that you should update by conditionalization so $P_2(D1) = P_1(D1|OD2)$.

By Bayes' Theorem $P_1(D1|OD2) = P_1(OD2|D1) \text{ x } P_1(D1)/P_1(OD2)$. By the law of total probability, $P_1(OD2) = P_1(OD2|D1) \text{ x } P(D1) + P_1(OD2|D2) \text{ x } P(D2) + P_1(OD2|D3) \text{ x } P(D3)$.

The assumption that Monty always opens an empty door and chooses randomly if he has a choice means that:

$P_1(OD2|D1) = 1/2$

$P_1(OD2|D2) = 0$

$P_1(OD2|D3) = 1$

Add the assumption that $P_1(D1) = P_1(D2) = P_1(D3) = 1/3$ and we get the standard answer that $P_2(D1) = 1/3$.

It was suggested in class that if Monty just randomly opened a door you didn't pick and it happened to be empty, you should still switch and I indicated that I thought this was incorrect. Now we have to be really careful here. We did learn that Monty opened door 2 of course, and it might really look as though this means that:

$P_1(OD2|D1) = 1/2$
$P_1(OD2|D2) = 1/2$
$P_1(OD2|D3) = 1/2$

Using these numbers, you would again get $P_2(D1) = 1/3$. However, we also learned that there was no prize behind door 2 and our notation suppresses this a bit. With the previous assumptions, OD2 means that of course door 2 is empty. Here, OD2 means that after we picked door 1, he opened door 2 and revealed that it was empty. But it might not have been empty! With the previous assumptions, the probability that it was empty was 1 so we need not explicitly factor this in to our calculations. But to be extra clear here, lets use the following notation:

OD2 = Monty opens door 2 after you have chosen door 1
~D2 = the prize is not behind D2

Now you have to conditionalize on everything you learned which in this case is OD2 & ~D2. With the previous assumptions, $P_1(OD2\ \&\ {\sim}D2) = P_1(OD2)$ but this is no longer true.

Now we have by Bayes Theorem $P_1(D1|OD2\&{\sim}D2) = P_1(OD2\&{\sim}D2|D1)$ x $P_1(D1)/P_1(OD2\&{\sim}D2)$.

By the law of total probability, $P_1(OD2\&{\sim}D2) = P_1(OD2\&{\sim}D2|D1)$ x $P(D1)$ + $P_1(OD2\&{\sim}D2|D2)$ x $P(D2)$ + $P_1(OD2\&{\sim}D2|D3)$ x $P(D3)$.

If Monty randomly opens one of the doors that you didn't pick we have $P_1(OD2\&{\sim}D2|D1) = 1/3$. This is because half the time that I choose door 1, he chooses door 2. Among those times, 2/3 of those times the prize will be somewhere other than door 2 and so ~D2 will be true.

Formally, $P_1(OD2\&{\sim}D2|D1) = P_1(OD2|{\sim}D2\&D1)$ x $P({\sim}D2) = 1/2$ x $2/3 = 1/3$.

This D1 case, plus the fact that D2 implies the falsity of ~D2 plus parallel reasoning in the D3 case gets you the following:

$P_1(OD2\&{\sim}D2|D1) = 1/3$
$P_1(OD2\&{\sim}D2|D2) = 0$
$P_1(OD2\&{\sim}D2|D3) = 1/3$

Now if you plug these numbers into Bayes Theorem above, you will get $P_2(D1)$ = 1/2.

Here is an older puzzle from Joseph Bertrand's *Calcul des probabilités* – the same book that introduces the chord problem.

> There are three boxes:
>
> 1. a box containing two gold coins
> 2. a box with two silver coins
> 3. a box with one gold and one silver coin
>
> You select a box at random and pick a random coin from that box. It is gold. What is the probability that the other coin in the box is also gold?

There is an obvious argument for the answer of 1/2. There were three boxes and now you know for sure that it is not in box 2. Therefore it is either in box 1 or box 3 and so the probability is 1/2. However, this is bad reasoning. In fact, the probability is 2/3 which you can calculate by using Bayes' Theorem. The key asymmetry is that the probability that you select a gold coin after picking box 1 is higher than if you pick box 3. Notice that if, for example, the coins were labeled 'coin 1' and 'coin 2' and you picked coin 1 and it was gold, now 1/2 would be the correct answer. This trick of labeling is related to the extremely confusing 'boy or girl paradox' which you can read about on wikipedia here: http://en.wikipedia.org/wiki/Boy_or_Girl_paradox

There are obvious formal features of this problem and others (like the 'three prisoner's problem') which make people say that 'Bertrand's box problem' is logically equivalent to the Monty Hall puzzle. I disagree. The numerical answers are the same for similar reasons, but they are not the same puzzle.